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LIQUID SLOSHING IN A SPHERICAL TANK  
FILLED TO AN ARBITRARY DEPTH

BY

WEN-HWA CHU

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8500 Culebra Road, San Antonio 6, Texas

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H. Norman Abramson, Director  
Department of Mechanical Sciences

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## FORWARD

The work presented in this report is the culmination of efforts on the part of the author that have extended over a period of two years, with partial support being received from several sources. The analysis was originally undertaken as part of a theoretical study under Contract DA-23-072-ORD-1251 sponsored by the Army Ballistic Missile Agency. The work was continued, and largely completed, under the program of internal research supported by Southwest Research Institute (Project 1059-2). Because of its relevance to experimental work already completed under the present program (NASA-MSFC Contract No. NAS8-1555) and published as Technical Report No. 2, it was felt highly desirable that this work should also be issued and distributed under auspices of this contract.

The material in this paper is also presented in Part I, Section 3, of a dissertation titled "Some Contributions to Unsteady Hydrodynamics in Engineering," submitted to the graduate faculty of The Johns Hopkins University in partial fulfillment for the requirements for the degree of Doctor of Philosophy.

# LIQUID SLOSHING IN A SPHERICAL TANK FILLED TO AN ARBITRARY DEPTH

## SUMMARY

The kernel function for liquid sloshing in a spherical tank filled to an arbitrary depth is shown to be related to the Green's function of the second kind and is constructed successfully by numerical means. Natural frequencies are then computed as eigen values of a matrix. Eigen functions are obtained at a finite number of points as the eigen vectors which are sufficient for approximate evaluation of the force acting on the container. Simple formulas of force and moment are given for both pitching and translational oscillation under a fixed gravitational field. Finally, comparisons of predicted natural frequencies and force response with experiments for a quarter-full tank are also given.

*Auth. abstr.*

## INTRODUCTION

Disturbances on a rocket or missile can induce sloshing of fuel in a partially filled tank. It in turn exerts excitation forces on the vehicle and in some cases can be detrimental to the trajectory or even results in loss of control. Sloshing in a circular cylindrical tank has been widely investigated with and without damping. To facilitate dynamic analysis, an equivalent mechanical model for circular tank is given in Reference 2. For a spherical tank, an ingenious semi-numerical method was given in Reference 1. However, the problem is only solved for three special cases, namely, nearly full, nearly empty, and half-full tanks. The restriction is due to the lack of the Green's function of the second kind (Neumann function) for the spherical bowl. Although the Green's function of the first kind for the spherical bowl is given in Reference 3, it is doubtful that a simple expression for the Green's function of the second kind exists in the toroidal coordinates, since the normal derivative on the spherical cap is a combination of two derivatives in this coordinate system. The sequence method given in Reference 4 is convergent for Green's function of the first kind but may diverge for the second kind. One may resort to Liouville-Neumann method (series method, Ref. 5) and prove it converges. But when the Green's function on the boundary

is desired, the kernel function is singular; thus it becomes increasingly more difficult to evaluate when more terms are needed. If we do not employ the Neumann function, an integral equation on the free surface is also obtained. Unfortunately, the eigen functions no longer satisfy the necessary orthogonal relationship (Ref. 6), thus they are the desired eigen functions only if the Neumann function is employed (Ref. 1). In this paper, a numerical scheme is devised to determine the desired kernel function, which is one component of the Neumann function, and then apply the same procedure as given in Reference 1 to evaluate the sloshing characteristics. Considerably more work is required to calculate the pressure on the wall, although in principle this can be done.

After the theory in the present paper was developed, some other approaches have been published. One approach (Ref. 7) seeks the variational solution based on Hamilton's principle through Rayleigh-Ritz method\*. Since only an integrated free surface condition was imposed, it is somewhat doubtful that accurate prediction of force response or pressure can be assured (Ref. 8), although error in the lowest mode frequency was less than one per cent for a flat cylindrical tank. In another approach (Ref. 9) finite difference techniques were employed to

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\* This method has been applied to spherical tank by Riley and Trembath whose results are shown in Figure 6.



seek eigen values in a boundary condition by three different methods. Method I and Method III (Ref. 9) use either Rayleigh quotient or Rayleigh-Ritz procedure, but are somewhat inferior (Ref. 9) to the Rayleigh-Ritz procedure applied to the continuous domain. Method II (Ref. 9) converts the problem into an equivalent matrix eigen value problem by eliminating the points outside the free surface through an inversion of matrix if the number of the other points is small, or through an influence coefficient type calculation if otherwise. In the latter case if there are  $N$  points on the free surface,  $N$  boundary value problems should be first solved (say by successive over-relaxation) before reduction to the eigen value problem of a  $N \times N$  matrix. Depending to a large extent on the number of net points required for a desired accuracy (say, 3 figures in frequencies and force response), the computing time (based on estimation on a GE 225 computer)\* of the last method for a spherical tank seems to be comparable to the present method. On the other hand, although further (significant) acceleration of the rate of convergence of the sub-routines in the present method in the present problem may be quite difficult, an alternative numerical scheme devised is expected to reduce

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\* It is estimated under the assumption that there are 20 free surface points and 300 total net points with 120 iterations for each boundary value problem (based on experience of a similar problem) and average speed for 5 multiplications, 4 additions, and one additional multiplication or division at each point in each iteration. There are other estimates based on experiences which yield approximately the same magnitude of computing time.

the computing time to one-half or further. Finally, Reference 10 has also been published in which the kernel function is constructed empirically, based on knowledge for half-full and full tank.

The purpose of the present paper is mainly to predict the natural frequencies and force response and to show how kernel functions are related to the Neumann function on the boundary and can be constructed numerically for a spherical tank. Analogous extension to other configurations or other problems may be possible but will not be treated in this paper.

# MATHEMATICAL FORMULATION

## A. Kernel Function

Let  $G(P, Q)$  and  $G_0(P, Q)$  be the Green's function of the second kind for the interior of the given spherical bowl (Fig. 1) and the sphere, respectively: (a) Both  $G(P, Q)$  and  $G_0(P, Q)$  possess continuous second derivatives and satisfy the Laplace equation inside the bowl and the sphere, respectively, except the point  $P=Q$ ; (b) Both  $G$  and  $G_0$  possess a unit sink,  $\frac{1}{4\pi R_{PQ}}$  at  $P=Q$  inside the bowl; (c)  $\frac{\partial G_0}{\partial n} = -\frac{1}{4\pi a^2}$  on the whole surface of the sphere,  $\frac{\partial G}{\partial n} = k_1 = \frac{-1}{A_R + A_F}$  on the surface of the bowl,  $R$  and  $F$ ; (d)  $G_0$  be that given in Reference 7;  $G(P, Q)$  satisfies the normalizing condition  $\int_{R+F} G(I, Q) dS_I = 0$  (Ref. 11). Following these conditions, it is well known (Ref. 11) that the Neumann function  $G$  is symmetric as well as  $G_0$ , i. e.,  $G(P, Q) = G(Q, P)$ ,  $G_0(P, Q) = G_0(Q, P)$ . When  $P, Q$  are both interior points, analogous to the proof of symmetric properties, one has

$$\begin{aligned}
 G(Q, P) - G_0(P, Q) &= \int_{R+F} \left\{ G_0(I, Q) \frac{\partial G(I, P)}{\partial n_I} - G(I, P) \frac{\partial G_0(I, Q)}{\partial n_I} \right\} dS_I \\
 &= k_1 \int_{R+F} G_0(I, Q) dS_I - \int_F G(I, P) \left[ \frac{\partial G_0(I, Q)}{\partial n_I} + \frac{1}{4\pi a^2} \right] dS_I \\
 &\quad - \int_{R+F} G(I, P) \frac{1}{4\pi a^2} dS_I \quad [1]
 \end{aligned}$$

which is an integral equation governing  $G(P, Q)$  where  $P, Q$  is inside the bowl, not on  $F$  and  $R$ .

For values of the Green's function with  $P, Q$  ( $P \neq Q$ ) both on  $F$ , not on  $R$ , apply directly the divergence theorem to the surface shown in Figure 2. Since there is an infinitesimal semi-sphere around the sinks at  $P$  and  $Q$  respectively, one finds

$$\frac{1}{2} G(Q, P) - \frac{1}{2} G_0(P, Q) = \int_{\substack{R+F \\ P \neq I \neq Q}} \left[ G_0(I, Q) \frac{\partial G}{\partial n_I} - G(I, P) \frac{\partial G_0(I, Q)}{\partial n_I} \right] dS_I \quad [2]$$

By making  $P$  and  $Q$  in Equation [1] approach  $P$  and  $Q$  on the free surface along its normal, Equation [1] can be reduced to Equation [2].

In Reference 1, for fuel sloshing in a spherical tank, only those eigen functions proportional to  $\cos \theta$  are needed: one shall see in the next section that it is sufficient to know one component  $H(\bar{P}, \bar{Q})$  of the Neumann function  $G(P, Q)$  to determine the sloshing characteristics.

Let

$$H(\bar{P}, \bar{Q}) = \frac{1}{\pi} \int_0^{2\pi} \int_0^{2\pi} G(P, Q) \cos \theta_P \cos \theta_Q d\theta_P d\theta_Q \quad [3a]$$

$$H_0(\bar{P}, \bar{Q}) = \frac{1}{\pi} \int_0^{2\pi} \int_0^{2\pi} G_0(P, Q) \cos \theta_P \cos \theta_Q d\theta_P d\theta_Q \quad [3b]$$

$$h_1(\bar{P}, \bar{Q}) = \frac{1}{\pi} \int_0^{2\pi} \int_0^{2\pi} [G(P, Q) - G_0(P, Q)] \cos \theta_P \cos \theta_Q d\theta_P d\theta_Q = H(\bar{P}, \bar{Q}) - H_0(\bar{P}, \bar{Q}) \quad [3c]$$

Since  $G$  and  $G_0$  are symmetric functions,  $H$ ,  $H_0$  and thus  $h_1$  are symmetric functions.

For points  $\bar{P}, \bar{Q}$  corresponding to P and Q respectively, inside the spherical bowl, Equation [1] can be integrated to yield

$$h_i(\bar{P}, \bar{Q}) = - \int_{\bar{F}} \mathcal{F}_0(\bar{I}, \bar{Q}) H_0(\bar{P}, \bar{I}) d\bar{S}_I - \int_{\bar{F}} \mathcal{F}_0(\bar{I}, \bar{Q}) h_i(\bar{P}, \bar{I}) d\bar{S}_I \quad [4]$$

for which the reversing of orders of integration are applied and can be justified by carrying out the details. The function F is defined by

$$\mathcal{F}_0(\bar{I}, \bar{Q}) = \frac{1}{\cos \theta_I} \int_0^{2\pi} \frac{\partial G_0(I, Q)}{\partial n_I} \cos \theta_Q d\theta_Q = \frac{1}{\pi} \int_0^{2\pi} \int_0^{2\pi} \frac{\partial G_0(r_I, r_Q; \psi_I, \psi_Q; \cos(\theta_I - \theta_Q))}{\partial n_I} \cos \theta_Q \cos \theta_I d\theta_Q d\theta_I \quad [5]$$

which is a nonsymmetric function as  $\frac{\partial G_0}{\partial n_I}$ .

Similarly, if both P and Q are on F, not on R, integration of Equation [2] yields

$$\begin{aligned} \frac{1}{2} \sqrt{\rho_P \rho_Q} h_i(\rho_P, \rho_Q) = & - \int_0^b [\sqrt{\rho_I \rho_Q} \mathcal{F}_0(\rho_I, \rho_Q)] [\sqrt{\rho_I \rho_P} H_0(\rho_P, \rho_I)] d\rho_I \\ & - \int_0^b [\sqrt{\rho_I \rho_Q} \mathcal{F}_0(\rho_I, \rho_Q)] [\sqrt{\rho_I \rho_P} h_i(\rho_P, \rho_I)] d\rho_I \end{aligned} \quad [6]$$

where  $\mathcal{F}_0$  and  $H_0$  are given in Appendices I and II. For  $\mathcal{F}_I = 0$ ,

$\mathcal{F}_0 = 0$ , almost everywhere on F, hence for half-sphere  $h_i(P, P) = 0$ ,  $H = H_0$  which is in agreement with Reference 1.

## B. Eigen Functions

The eigen functions  $\phi_n$  are assumed to possess the following properties: (a)  $\phi_n$  is regular inside the bowl and  $\nabla^2 \phi_n = 0$

(b)  $\frac{\partial \phi_n}{\partial n_I} = 0$  on R,  $\frac{\partial \phi_n}{\partial n_I} = \tilde{\lambda}_n \phi_n(I)$  on F, (c)  $\phi_n(P) = \varphi_n(\bar{P}) \cos \theta_P$  (Ref. 1).

The last condition is appropriate for translational oscillation of the tank.

Analogous to Equations [1] and [2]

$$\phi_n(P) = \int_F G(I, P) \tilde{\lambda}_n \phi_n(I) dS_I \quad [7]$$

when P is inside the spherical bowl,  $R + F$ .

$$\frac{1}{2} \phi_n(P) = \int_F G(I, P) \tilde{\lambda}_n \phi_n(I) dS_I \quad [8]$$

when P is on F.

Analogous to Equations [4] and [5] by integration

$$\psi_n(\bar{P}) = \int_{\bar{F}} H(\bar{I}, \bar{P}) \psi_n(\bar{I}) d\bar{S}_I, \quad \bar{P} \text{ not on } \bar{F}. \quad [9]$$

$$\frac{1}{2} \psi_n(\bar{P}) = \tilde{\lambda}_n \int_0^b H(\bar{P}, \bar{P}') \psi_n(\bar{P}') \bar{P}' d\bar{P}', \quad \bar{P} \text{ on } \bar{F}. \quad [10]$$

This shows only  $H(\bar{P}, \bar{Q})$  is needed for the pertinent eigen functions.

### C. Sloshing Force and Pressure in Translational Oscillation

By introducing a displacement potential relative to the tank

$\Psi_d = \sum a_n(t) \phi_n(r, \psi, \theta)$ , the sloshing force acting on the container is derived from the Lagrangian's equation in Reference 1, namely

$$F_s = -M_L \ddot{U} - \rho_L \sum \beta_n \ddot{a}_n \quad [11]$$

where  $M_L = \text{mass of the liquid} = \rho_L \frac{\pi}{3} [2a^3 + 3a^2 z_F - z_F^3]$

$$\beta_n = \frac{\omega_n^2}{g} \int_F x \phi_n dS = \pi b^2 \Omega_n \frac{b}{a} \int_0^b \phi_n^2(\rho_I) \rho_I^2 d\rho_I \quad [11a]$$

$$\alpha_n = \int_F \phi_n^2 dS = \pi b^2 \int_0^b \phi_n^2(\rho_I) \rho_I d\rho_I \quad [11b]$$

$$\ddot{a}_n = \frac{\frac{g}{\omega_n^2} \cdot \frac{\beta_n}{\alpha_n} \ddot{U}}{\frac{\omega_n^2}{\omega^2} - 1} \quad [11c]$$

The velocity potential

$$\phi_v = \sum_{n=1}^{\infty} \dot{a}_n(t) \phi_n(r, \psi, \theta) + \dot{U} x \quad [12]$$

The pressure on the container

$$p \cong -\rho \sum \ddot{a}_n(t) \phi_n(a, \psi, \theta) - \rho \ddot{U} x \cong -\rho \frac{\partial \phi_v}{\partial t} \quad [13]$$

within the accuracy of the linearized theory. Equation [11] can also be obtained directly by integration of pressure (Appendix V).

Once  $\phi_n$  on F is evaluated, one may employ  $G_o$  to obtain  $\phi_n(P)$  from

$$\phi_n(P) = \int_{R+F} \left[ G_o(I, P) \frac{\partial \phi_n}{\partial n_I} - \phi_n(I) \frac{\partial G_o(I, P)}{\partial n_I} \right] dS_I = \varphi_n(\bar{P}) \cos \theta_p \quad [14a]$$

$$\varphi_n(\bar{P}) = \int_F H_o(\bar{I}, \bar{P}) \tilde{\lambda}_n \varphi_n(I) d\bar{S}_I - \int_F \tilde{J}_o(\bar{I}, \bar{P}) \varphi_n(I) d\bar{S}_I \quad [14b]$$

The integral on R dropped out as  $\frac{\partial \phi_n}{\partial n_I} = 0$  on R and  $\frac{\partial G_o}{\partial n_I} = \text{constant}$  on R.

For P on R, not on F, the integrands of the integrals in Equation [14b] are nonsingular, hence  $\varphi_n(\bar{P})$  can be calculated by well-known numerical methods. For contact points both on R and F, the value of  $\varphi_n(\rho_n)$  may be obtained by evaluation of the integral by midpoint formula.

#### D. The Moment Under Translational Oscillation

For translational oscillations, the velocity potential is proportional to  $\cos \theta$  and the flow is antisymmetric. It produces a horizontal force  $F_s$  in the x-direction and a couple  $C_s$  about the center of the tank (Fig. 3a). There is no moment around z axis or x axis by symmetry. The moment about a fixed point O' on the z axis is

$$M_l = F_s (l - l_x) + C_s \quad [15a]$$

It is not necessary to determine  $l_x$  when the force  $F_s$  and the moment  $M_l$  are the desired information in dynamic problems. For a sphere, all the pressure forces acting on the shell passes through its center, hence produces no moment about it, i. e. ,

$$M_{l=0} = -F_s l_x + C_s = 0 \quad [15b]$$

Therefore the moment about O' is simply

$$M_l = F_s l \quad [15c]$$

This statement can be easily shown by integration of the moments due to pressure on the wall.

#### E. Pitching Oscillation

Consider a pitching oscillation of amplitude  $\alpha_y$  around an axis which is parallel to y axis and at a vertical distance  $l$  below y axis (Fig. 3b). In Figure 3b, it is clear that



$$\cos \lambda = \frac{z+l}{\sqrt{(z+l)^2 + x^2}}, \quad \sin \lambda = \frac{x}{\sqrt{(z+l)^2 + x^2}}$$

The radial distance of any point (x, y, z) from the axis of rotation is  $\sqrt{(z+l)^2 + x^2}$ . The velocity components on the sphere due to rotation are:

$$u_s = (\sqrt{(z+l)^2 + x^2} \dot{\theta}_y) \cos \lambda = (z+l) \dot{\theta}_y$$

$$w_s = -(\sqrt{(z+l)^2 + x^2} \dot{\theta}_y) \sin \lambda = -x \dot{\theta}_y$$

The boundary condition on the wetted sphere, R, is

$$\begin{aligned} \left( \frac{\partial \phi}{\partial r} \right)_{r=a} &= \left[ u_s \frac{x}{r} + v_s \frac{y}{r} + w_s \frac{z}{r} \right]_{r=a} \\ &= \left[ (z+l) \dot{\theta}_y \sin \psi \cos \theta - \sin \psi \cos \theta x \dot{\theta}_y \right]_{r=a} \\ &= \dot{\theta}_y l (\sin \psi \cos \theta) \end{aligned}$$

This is equivalent to a translational oscillation of amplitude  $U = l \dot{\theta}_y$  in the direction of x. Since the boundary condition on the free surface is the same in the presence of a fixed gravitational field, the result for translational oscillation can be applied. There is an additional static tipping force which can be obtained by integrating the additional static pressure  $p'$  over R (Ref. 14)

$$p' = \rho g x \theta_y \quad [16a]$$

$$\begin{aligned} F'_s &= \theta_y \rho g \iint_R x \cos(n, x) dS \\ &= \theta_y \rho g \iiint_{V_L} \text{div}(ix) dV = \theta_y \rho g V_L = M_L g \theta_y \end{aligned} \quad [16b]$$

This force acts along an x axis rotating with the tank.

Similarly, there is an additional moment

$$\begin{aligned} M'_S &= \int_R p' \cos(n, x) (z+l) ds - \int_R p' \cos(n, z) x ds \\ &= \rho g \theta_y \pi a^3 l \left[ \frac{2}{3} + \cos \psi_F - \frac{1}{3} \cos^3 \psi_F \right] = M_L g \theta_y l \end{aligned} \quad [16c]$$

The total force along an axis x rotating with the tank is

$$F_x \cong F_S + F'_S \quad [16d]$$

The moment about O' is approximately

$$M_O = F_S l + M'_S = F_x l \quad [16e]$$

The total force in the horizontal direction is still  $F_S$ . When there is tank fixed axial acceleration, the method of superposition presented in Reference 14 can be used to determine the x-force.

An equivalent mechanical model for sloshing in spherical tank is given in Reference 15, but unfortunately the extrapolation to include damping was not as successful as in the case of a cylindrical tank (Ref. 2) and could only be used for order of magnitude estimates (Ref. 15).

## NUMERICAL METHOD

### A. Approximate Determination of the Kernel Function at a Finite Number of Points

Numerical quadrature formula will be used to replace the integral Equation [6] by a matrix equation. There is a minor difficulty due to the presence of logarithm's singularity at  $\bar{P} = \bar{I}$  or  $\rho_i = \rho_i$ , the latter of which is the integration variable. In the original manuscript, an attempt was made to devise a more sophisticated quadrature formula, expecting higher accuracy. Unfortunately, it seems to contain integrals difficult to express in known functions, or require very careful process of taking limit under the integral signs. Further, the apparent higher order terms may be actually very large and not negligible. To reduce total effort, the present numerical scheme based on midpoint formula is devised.

The integrals are divided into N equal parts (N=20 will be used) and the field point is one of the centers of the intervals. A simple midpoint formula will not be applicable when the logarithmic singularity appears at the midpoint, but if the interval is subdivided into four intervals (or more) the error may become acceptable. For example, consider the integral

$$\begin{aligned}
 S &= \int_{\rho_i - \frac{\Delta}{2}}^{\rho_i + \frac{\Delta}{2}} \ln |\rho_i - \rho| d\rho = \int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} \ln |\rho| d\rho \\
 &= \Delta \ln\left(\frac{\Delta}{2}\right) - \Delta = \Delta \ln \Delta - 1.69315 \Delta
 \end{aligned}
 \tag{17}$$

With four equal subintervals, the midpoint formula yields

$$\begin{aligned} S &= \left[ 2 \ln\left(\frac{1}{8}\Delta\right) + 2 \ln\left(\frac{3}{8}\Delta\right) \right] \frac{\Delta}{4} = \left[ \Delta \ln \Delta + \left(\frac{1}{2} \ln 3 - \ln 8\right) \Delta \right] \\ &= \Delta \ln \Delta - 1.5301 \Delta \end{aligned} \quad [18]$$

The error is  $.16 \Delta$ . For  $\Delta = 1/20$  ( $N=20$ ,  $b=1$ ), the relative error is less than 0.18%.

Let  $b=1$ ,

$$g_{i,j} = h_i(\rho_i, \rho_j) \sqrt{\rho_i \rho_j} \quad [19a]$$

$$H_{ij}^{(0)} = \sqrt{\rho_i \rho_j} H_0(\rho_i, \rho_j) \quad [19b]$$

$$F_{ij}^{(0)} = F_{ji} = \sqrt{\rho_i \rho_j} F_0(\rho_i, \rho_j) \quad (\text{note the order of } i \text{ \& } j) \quad [19c]$$

then Equation [6] can be rewritten as

$$\frac{1}{2} g_{i,j} = - \int_0^1 F_{i,l} H_{lj}^{(0)} d\rho_l - \int_0^1 F_{j,l} g_{i,l} d\rho_l \quad [20]$$

Two similar numerical schemes will be presented. The first scheme was actually employed in the example, while the second scheme is the alternative scheme requiring much less computer time. In the first scheme one evaluates  $F_{ij}^{(0)}$  and  $H_{ij}^{(0)}$  at  $N \times 4N$  points, assuming four point midpoint formula.  $i^{\text{th}}$  point on the free surface is located at the midpoint of the  $i^{\text{th}}$  interval ( $i=1,2, \dots, N$ ).  $j$  represents the integration variable located at the midpoint of the subinterval ( $j=1,2, \dots, 4N$ ).

Thus

$$\rho_i = \frac{1}{2N} + \frac{(i-1)}{N} = \frac{\Delta}{2} + (i-1) \Delta \quad (b=1, i=1, 2, \dots, N) \quad [21a]$$

$$\rho_j = \frac{1}{8N} + \frac{(j-1)}{4N} = \frac{\Delta}{8} + (j-1) \frac{\Delta}{4} \quad (b=1, j=1, 2, \dots, 4N) \quad [21b]$$

To describe the second (alternate) scheme, consider the whole square domain to be composed of  $N \times N$  square subdomains. In all the diagonal squares,  $\mathcal{F}_{ij}^{(0)}$  and  $H_{ij}^{(0)}$  are evaluated as in the first scheme (four values in each square), but they will take the value of the functions at the center of each square in off-diagonal domain, which are also evaluated. These data will be denoted by  $\tilde{\mathcal{F}}_{ij}^{(0)}$ ,  $\tilde{H}_{ij}^{(0)}$ . Total number of evaluation (both  $\mathcal{F}_{ij}^{(0)}$  and  $H_{ij}^{(0)}$ ) are  $N \times 4N$  in the first scheme, but  $N \times N + 3N$  in the second scheme (4 point midpoint formula for diagonal integral). For instance, consider

$$M_{i,l} = \int_0^1 \mathcal{F}_{f,l} H_{i,j}^{(0)} d\rho_f$$

In the first scheme,  $\rho_i, \rho_j$  being given by Equations [21a], [21b],

$$\begin{aligned} M_{i,l} &= \sum_{k=1}^N \int_{k\Delta - \frac{\Delta}{2}}^{k\Delta + \frac{\Delta}{2}} \mathcal{F}_{f,l} H_{i,j}^{(0)} d\rho_f \\ &\approx \sum_{k=1}^N \sum_{j=4k-3}^{4k} \frac{\Delta}{4} \mathcal{F}_{f,l} H_{i,j}^{(0)} \\ &= \sum_{j=1}^{4N} \frac{\Delta}{4} \mathcal{F}_{f,l} H_{i,j}^{(0)} = \sum_{j=1}^{4N} \tilde{\mathcal{F}}_{f,l}^{(0)} \tilde{H}_{i,j}^{(0)} \end{aligned} \quad [22]$$

In the second scheme, first let  $p_j = j\Delta$ ,  $p_i = p_l = p_k$  given by Equation [21a]

$$M_{i,l} = \sum_{k=1}^N \int_{k\Delta - \frac{\Delta}{2}}^{k\Delta + \frac{\Delta}{2}} f_{i,l} H_{ij}^{(0)} dp_j \approx \sum_{k=1}^N S_{k,i,l} \quad [23_0]$$

$$S_{k,i,l} \approx \sum_{j=k-\frac{3}{8}, k-\frac{1}{8}, k+\frac{1}{8}, k+\frac{3}{8}} \frac{\Delta}{4} f_{i,l} H_{ij}^{(0)}, \quad i=l=k \quad [23a]$$

$$S_{k,i,l} \approx f_{k,l} \sum_{j=k-\frac{3}{8}, k-\frac{1}{8}, k+\frac{1}{8}, k+\frac{3}{8}} \frac{\Delta}{4} H_{ij}^{(0)}, \quad l \neq k, i=k \quad [23b]$$

$$S_{k,i,l} \approx H_{ik}^{(0)} \sum_{j=k-\frac{3}{8}, k-\frac{1}{8}, k+\frac{1}{8}, k+\frac{3}{8}} \frac{\Delta}{4} f_{j,l}, \quad i \neq k, l=k \quad [23c]$$

$$S_{k,i,l} \approx \Delta H_{ik}^{(0)} f_{k,l}, \quad i, l \neq k \quad [23d]$$

Equations [23a] — [23d] can be condensed into the single formula with

$p_j$  given by Equation [21b] and one finds

$$A_i \approx \sum_{j=1}^{+N} \frac{\Delta}{4} f_{ij}^{(0)} H_{ij}^{(0)} \quad [23]$$

Similarly, the second integral on the right side of Equation [20] is

$$\int_0^1 f_{i,l} g_{ij} dp_j \approx \sum_{k=1}^N C_{ik} D_{kl} \quad [24]$$

where

$$C_{ik} = g_{ik} \quad [24a]$$

$$D_{kl} \approx \sum_{j=k-3}^{4k} \frac{\Delta}{4} \mathcal{F}_{j,l} = \frac{\Delta}{4} \sum_{j=k-3}^{4k} \mathcal{F}_{l,j}^{(0)} \quad [24b]$$

In the second scheme, replace  $\mathcal{F}_{l,j}^{(0)}$  by  $\tilde{\mathcal{F}}_{l,j}^{(0)}$  and  $H_{ij}^{(0)}$  by  $\tilde{H}_{ij}^{(0)}$ .

The integral Equation [6] is therefore reduced approximately to the matrix equation,

$$\frac{1}{2} [C] = - [M] - [C] [D] \quad [25]$$

the solution of which is

$$[C] = - [M] \left\{ \frac{1}{2} [I] + [D] \right\}^{-1} \quad [25a]$$

[I] is a unit matrix. [C], [M], [D] are square matrices of which the elements of  $i^{\text{th}}$  row and  $l^{\text{th}}$  column are  $C_{il}$ ,  $M_{il}$ ,  $D_{il}$ , respectively. As a check of accuracy, the symmetric property  $C_{ij} \approx C_{ji}$  should hold approximately. Then we can use the average value for the corrective term in the kernel function, i. e.,

$$\sqrt{\rho_i \rho_j} h_i(\rho_i, \rho_j) \approx \tilde{C}_{ij} = \frac{1}{2} [C_{ij} + C_{ji}] \quad [26]$$

The kernel function is therefore

$$\sqrt{\rho_i \rho_j} H(\rho_i, \rho_j) = \sqrt{\rho_i \rho_j} H_i(\rho_i, \rho_j) + \tilde{C}_{ij} \quad [27]$$

$\tilde{C}_{ij}$  being known at discrete points corresponding to both  $\rho_i, \rho_j$  given by equation [21a]. The difficulty of the problem, however, lies in the accurate and rapid evaluation of the function  $\mathcal{F}_{ij}^{(0)}$  and  $H_{ij}^{(0)}$ .

B. Determination of Eigen Vectors,  $\varphi^{(n)}$  and Natural Frequencies

The eigen function takes the value  $\varphi_n(\rho)$  on F, which is governed by

$$\varphi_n(\rho) = \frac{2\Omega_n}{a} \int_0^b [H_0(\rho, \rho') + h_1(\rho, \rho')] \varphi_n(\rho') \rho' d\rho' \quad [28]$$

where

$$\Omega_n = \frac{\omega_n^2 a}{g}, \quad g \text{ being effective gravitational acceleration} \quad [28a]$$

Let  $\varphi^{(n)}(\rho) = \sqrt{\rho} \varphi_n(\rho)$ , then

$$\varphi^{(n)}(\rho) = \Omega_n \cdot \frac{2}{a} \int_0^b [\sqrt{\rho\rho'} H(\rho, \rho')] \varphi^{(n)}(\rho') d\rho' \quad [29]$$

Analogous to Section A, the matrix approximation of Equation [29] is

$$\left( \frac{N}{2\Omega_n \frac{b}{a}} \right) \{ \varphi^{(n)} \} = [A] \{ \varphi^{(n)} \} \quad [30]$$

where the factor 1/2 on the right-hand side is in agreement with Reference 1, since the strength of the Green's function has not been doubled in this paper. The elements of the matrix A is

$$A_{ik} \cong \tilde{H}_{ik}^{(0)} + \tilde{C}_{ik}, \quad i \neq k \quad [30a]$$

$$A_{ik} \cong \frac{1}{4} \sum_{j=1}^{4k} H_{ij}^{(0)} + \tilde{C}_{ik} = \tilde{H}_{ik}^{(0)} + \tilde{C}_{ik}, \quad i = k \quad [30b]$$

where i, k corresponds to  $\rho_i, \rho_k$  both given by Equation [21a] and both vary from 1 to N.



In the first scheme,  $H_{ik}^{(0)}$  is not evaluated at the center of any square subdomain to reduce computing time and is approximated by

$$H_{ik}^{(0)} = \frac{1}{2} \left[ \frac{1}{4} \sum_{j=4k-3}^{4k} H_{ij}^{(0)} + \frac{1}{4} \sum_{j=4i-3}^{4i} H_{kj}^{(0)} \right] \quad (i, k = 1 \text{ to } N) \quad [30c]$$

where

$$H_{kj}^{(0)} = H_{jk}^{(0)} \quad (\text{property of symmetry})$$

In the second scheme, replace  $H_{ij}^{(0)}$  by  $\tilde{H}_{ij}^{(0)}$  ( $i = 1$  to  $N$ ,  $j = 1$  to  $4N$ ).

The largest eigen values  $\frac{N}{2 \Omega_n \frac{b}{a}}$  of Equation [30] yields the least resonant frequency parameters.  $\Omega_n$ . And the eigen vectors will be employed in evaluation of the force response.

### C. Evaluation of Force

The sloshing force for translational oscillation is

$$F_s = -M_L \ddot{U} - \rho_L \ddot{U} \sum_{n=1}^{\infty} \frac{a \beta_n \left( \frac{\beta_n}{a n} \right)}{\Omega_n (\frac{\omega_n^2}{\omega^2} - 1)} \quad [31]$$

where

$$M_L = \rho_L \frac{\pi}{3} [2a^3 + 3a^2 z_F - z_F^3] \quad ; \quad M_L^* = \frac{M_L}{\rho_L a^3} = \frac{\pi}{3} [2 + 3 \frac{z_F}{a} - \frac{z_F^3}{a^3}] \quad [31a]$$

$$\alpha_n = \pi b^2 \int_0^1 |\psi^{(n)}(\rho)|^2 d\rho = \pi b^2 \alpha_n^* \triangleq \pi b^2 \sum_{i=1}^N \varphi_i^{(n)2} \Delta \quad [31b]$$

$$\beta_n = \Omega_n \left( \frac{b}{a} \right) b^2 \pi \int_0^1 \psi^{(n)}(\rho) \rho^{3/2} d\rho = \pi b^2 \beta_n^* = \pi b^2 \Omega_n \left( \frac{b}{a} \right) \sum_{i=1}^N \varphi_i^{(n)} \rho_i^{3/2} \Delta \quad [31c]$$

The nondimensional force

$$F_s^* = \frac{F_s}{\rho_2 a^2 g U} = M_1^* \frac{\omega^2 a}{g} + \frac{\omega^2 a}{g} \sum_{n=1}^{\infty} \frac{\pi (\frac{b^2}{a^2}) \beta_n^{*2} / \alpha_n}{\Omega_n (\frac{\omega_n^2}{\omega^2} - 1)} \quad [31']$$

#### D. Precision Problem

The functions  $\mathcal{F}_0$ ,  $H_0$  have been first expressed in terms of complete elliptical integrals of the first kind, the second kind, and the third kind and of simple elementary functions (Appendix I, II). The elliptical integrals of the third kind are expressed in terms of Heuman's Lamda function  $\Lambda_0$  (Ref. 12), which is again expressed either in a series form or in a close form of incomplete and complete elliptical integrals of the first and second kind, i. e.,

$$\Lambda_0(\beta, k) = \frac{2}{\pi} [E(k)F(\beta, k') + K(k)E(\beta, k') - K(k)F(\beta, k')] \quad [32]$$

In  $\mathcal{F}_3$  and  $H_{03}$ , a serious precision problem occurs due to almost complete loss of significant figures in subtractions for  $\beta_i, \rho_i$  both small. At first, the series form of the Lamda function was used, but it was found that the series is very slowly convergent when the parameter is near unity, especially if double precision or twelve significant figures are sought. Then it is resorted to the iterative methods for evaluating elliptic integrals (Ref. 13), which converges to  $10^{-9}$  within four or five iterations. Although the complete elliptic integrals can be computed very rapidly, the subroutine NEFF (Appendix III) for incomplete integrals and a difference related to it

consumes 8 seconds (used twice), while the total time for evaluating  $\mathcal{F}_0$ ,  $H_0$  is only 25 seconds at each point, all on the GE 225 computer. Longer time would be required for higher precision as the number of iterations increases.

To increase the precision, analytic subtractions are made so that no significant subtraction remains, if possible. Noniterative subtractions in which four or less figures are lost are acceptable if four or more significant figures out of eight (single regular precision on the machine) is desired. The technique can be illustrated by the following cases:

- (1) Let  $(A - B)$ , the difference of  $A$  and  $B$  is small but can be expressed analytically without subtraction. Then, for example,  $\frac{1}{\sqrt{A}} - \frac{1}{\sqrt{B}}$  should be evaluated from  $\frac{-1}{\sqrt{A}\sqrt{B}} \frac{(A-B)}{\sqrt{A} + \sqrt{B}}$   
e. g. ,  $A = 2$  ,  $B = 2 + \delta$  ,  $\delta \ll 1$  ,  $(A-B) = \delta$
- (2) Let  $k_n$ 's be small (positive) quantities containing no subtraction, then  $(1+k_1)(1+k_2) \dots (1+k_n) - 1$  should be evaluated by repeated application of the simple relation that
 
$$(1+k_1)(1+k_2) - 1 = k_1 + k_2 + k_1 k_2$$
- (3) To subtract a desired quantity from a known function may require a new subroutine for this function performing significant subtraction analytically, e. g., NEFF. (Appendix III)

Aside from relatively mechanical operations, the device of DKEF and NEFF subroutines, the following relation was expedient (Appendix IV).

$$\pi_N = -\alpha^2 \left\{ \frac{\pi}{2} \frac{(1 - \frac{1}{2}\alpha^2) L_{id}}{\sqrt{-\alpha^2(1-\alpha^2)(k^2-\alpha^2)}} - \frac{\frac{1}{2}K(k)}{(1-\alpha^2)} \right\} \quad [33]$$

$\pi_N$  and  $L_{id}$  are defined by Equations [IV-6], [IV-8], respectively.

It is noted that, after a small manipulation, direct numerical integration of the integrals  $F_3, H_{03}$  at sampling points of the entire domain of  $\rho_i, \rho_t$  was also computed by Weddle's rule. Although four or more significant figures can be obtained, it is deemed too slow over the major part of the domain. For instance it took about 5 and 2-1/2 minutes respectively for  $F_3$  and  $H_{03}$  on a GE 225 computer with 384 intervals, or a relative error of about  $10^{-5}$  otherwise at a point near the right lower corner of the domain ( $\rho_i, \rho_t$  near unity). These values at sampling points are valuable as they serve as a good check on the present computer program, which evaluates  $F_1, F_2, F_3, H_{01}, H_{02}, H_{03}$ , all together at a rate of 25 seconds per net point (on the same computer with an accuracy of four or more significant figures).

# EXAMPLE: FUEL SLOSHING IN A QUARTER-FULL TANK

First,  $J_0$  and  $H_0$  are generated, then the matrix Equation [25] is solved. The corrective part  $C_{ij}$  to the kernel function obtained is symmetric almost to four figures (Table I). The relative errors in the sample points are less than 0.3% or better. Since these values are quite representative, the values of  $C_{ij}$  at other points are not shown in the table.

Next, the eigen values and eigen vectors of Equation [30] and then the force response of Equation [31] are calculated. The calculated first four eigen values are 9.48863, 2.0591201, 1.2003387, 0.84773955, respectively. The corresponding frequency parameters are compared with experiments in Figures 4a and 4b. It seems that the values are well within possible experimental error, although it may be slightly less than the actual value, noting that natural frequencies are somewhat smaller for larger amplitudes of oscillation.

The constants needed to calculate the force response are compared with graphical values given by Reference 1 in Table II. Since the coefficient  $D^2/C$  is in agreement with Budiansky's value, the main difference lies in the value of first natural frequency for frequency range in its neighborhood. Since graphically interpolated value is less reliable, which is also confirmed experimentally in this case, only the present theory is compared with experiments (Ref. 15) in Figure 5. The difference between theory and experiments, perhaps, is essentially due to finite amplitude effect. But the agreement seems to be quite reasonable.

i/j	1	2	9	10	11	19	20
1	1.0683901 $\times 10^{-5}$	5.5419177 $\times 10^{-5}$		8.2274045 $\times 10^{-4}$		1.8776441 $\times 10^{-3}$	1.9692882 $\times 10^{-3}$
2	5.5416096 $\times 10^{-5}$	2.8747310 $\times 10^{-4}$		4.2729657 $\times 10^{-3}$		9.7666734 $\times 10^{-3}$	1.0244200 $\times 10^{-2}$
9			4.8220011 $\times 10^{-2}$	5.6585628 $\times 10^{-2}$	6.5237782 $\times 10^{-2}$		
10	8.2212278 $\times 10^{-4}$	4.2698876 $\times 10^{-3}$	5.6576890 $\times 10^{-2}$	6.6504031 $\times 10^{-2}$	7.6811489 $\times 10^{-2}$	1.6376404 $\times 10^{-1}$	1.7254674 $\times 10^{-1}$
11			6.5216369 $\times 10^{-2}$	7.6797981 $\times 10^{-2}$	8.8875807 $\times 10^{-2}$		
19	1.8737714 $\times 10^{-3}$	9.7465323 $\times 10^{-3}$		1.6341799 $\times 10^{-1}$		5.6357121 $\times 10^{-1}$	6.3742676 $\times 10^{-1}$
20	1.9743665 $\times 10^{-3}$	1.027057 $\times 10^{-3}$		1.7294743 $\times 10^{-1}$		6.3772697 $\times 10^{-1}$	7.6960677 $\times 10^{-1}$

TABLE I: Samples of Corrective Part,  $C_{ij}$  for Quarter-Full Tank

n	$\Omega_n = \frac{\omega_n^2 a}{g}$		$D_n^2 / C_n = \left( \frac{\beta_n}{\Omega_n b/a} \right)^2 / \alpha_n^*$	
	Chu	Budiansky (Fig. 9, Ref. 1)	Chu	Budiansky (Fig. 10, Ref. 1)
1	1.2169314	1.1323	0.24919	0.249586
2	5.6077376	5.4443	$0.44687 \times 10^{-3}$	$0.4172 \times 10^{-3}$
3	9.6197893	9.2325	$0.44758 \times 10^{-4}$	$0.7664 \times 10^{-4}$
4	13.620935	---	$0.112633 \times 10^{-4}$	---

TABLE II: Comparison of Constants with Data  
from Reference 1

## CONCLUSIONS AND DISCUSSIONS

The present theory and computer program seem to yield satisfactory predictions of natural frequencies and force response in comparison with experiments for a quarter-full spherical tank. The computer program is expected to be applicable to other liquid depths, although not beyond improvement in efficiency. The results also confirm the theory that the kernel function is related to the Neumann function on the boundary and that this function can be constructed by adding a corrective part to a known Green's function numerically for practical applications. Extensions to other problems may be possible, but one must resolve the precision problem if it exists and one may also find a more sophisticated numerical scheme to be more desirable, either in accuracy or in efficiency.



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## NOMENCLATURE

$a$  = radius of the spherical tank

$A_F$  = area of undisturbed free surface

$A_R$  = area of wetted surface of sphere

$b$  = maximum value of  $\rho$ , radius of free surface

$d$  = tank diameter,  $2a$

$$D(q) = \frac{(E(q) - K(q))}{q^2} \quad (\text{c.f. Appendix III})$$

DKEF = a function in the computer program (c.f. Appendix III)

$F$  = the undisturbed free surface

$F_s$  = horizontal force acting on the tank due to fuel sloshing

$$\mathcal{F}_{ij} = \sqrt{\rho_i \rho_j} \mathcal{F}_0(\rho_i, \rho_j) = \mathcal{F}_{ji}^{(0)} \quad (\text{note the order of } \rho_i, \rho_j \text{ in } \mathcal{F}_{ij})$$

$\mathcal{F}_0(\rho, \rho')$  = integrated kernel function related to

$g$  = effective gravitational acceleration

$$g_{ij} = \sqrt{\rho_i \rho_j} h_i(\rho_i, \rho_j)$$

$G(P, Q)$  = Green's function of the second kind for the spherical bowl

$G_0(P, Q)$  = Green's function of the second kind for a sphere

$h(P, Q)$  = additional part of Green's function for spherical tank other than half-full

$h_i(\rho, \rho')$  = integrated kernel function related to  $h(P, Q)$

$H(\rho, \rho')$  = integrated kernel function related to  $G(P, Q)$

$H_0(\rho, \rho')$  = integrated kernel function related to  $G_0(P, Q)$

$$H_{ij} = \sqrt{\rho_i \rho_j} H(\rho_i, \rho_j)$$

$$H_{ij}^{(0)} = \sqrt{\rho_i \rho_j} H_0(\rho_i, \rho_j)$$

- $I$  = point of integration, except  $[I]$  being the unit matrix  
 $K(Q), E(Q)$  = complete elliptic integrals of first and second kind, respectively  
 $M_L$  = total mass of liquid (fuel)  
 $n$  = outer normal  
 NEFF = a function in the computer program (c.f. Appendix III)  
 $P(Q) = (K(Q) - \frac{\pi}{2})$  (c.f. Appendix III)  
 $\bar{P}(r, \psi)$  = a ring corresponding to  $P(r, \psi, \theta)$   
 $q_1 = \frac{2\sqrt{pp'}}{p+p'}$   
 $q_2 = \frac{2a\sqrt{pp'}}{\sqrt{(pp'-b^2)^2 + z_F^2(p-p')^2 + 4pp'a^2}}$   
 $\bar{Q}, \bar{I}$  analogous to  $P$  but related to  $Q$  and  $I$ , respectively  
 $Q_s$  defined by Equation [I-10a]  
 $r, \psi, \theta$  = spherical coordinates  
 $R$  = the wetted spherical surface before sloshing unless defined by [I-2]  
 $R(P, P') = \sqrt{r'^2 + \frac{a^4}{r^2} - 2\frac{r'}{r}a^2\cos\theta}$   
 $R_{za}, R_{\theta}, R_{\psi}$  defined by Equations [I-10h], [I-10b], [II-5b], respectively  
 $R_{PQ}$  = distance between the points  $P$  and  $Q$   
 $dS$  = element of surface  
 $d\bar{S} = dS/d\theta \rightarrow p dp$  on  $F$   
 $U$  = horizontal displacement of container in the x-direction  
 $x = r \cos \theta$   
 $z_F$  = vertical distance of free surface from center of sphere; positive upward  
 $\alpha_n$  defined by Equation [12b],  $\int_F \phi_n^2(I) dS_I$

$\beta_n$  defined by Equation [12a]

$\cos \gamma$  = angle between the vectors  $\underline{OP}$  and  $\underline{OP'}$

$$\lambda_1, \lambda_2 = \frac{1}{\rho \rho'} [-z_F^2 \pm \sqrt{z_F^4 + z_F^2(\rho^2 + \rho'^2) + \rho^2 \rho'^2}] \quad , \text{ respectively}$$

$\Lambda_0(\psi, \varrho)$  = Heuman's lambda function (Ref. 9)

$$\tilde{\lambda}_n = \frac{\omega_n^2}{g}$$

$\Pi(\alpha^2, \varrho)$  = complete elliptic integrals of the third kind (Ref. 9)

$\rho$  = radial distance from a point on the free surface to the center of the free surface

$\rho', \rho_i$  =  $\rho$  of integration variable

$\rho_L$  = density of liquid (fuel)

$\phi$  = velocity potential,  $\nabla \phi = \underline{q}$  ,  $\underline{q}$  being the velocity vector

$\phi_n$  = nth eigen function

$\psi_n$  = nth integrated eigen function related to  $\phi_n$

$$\psi^{(n)} = \sqrt{\rho} \psi_n(\rho)$$

$\omega$  = frequency of oscillation

$\omega_n$  = nth resonant frequency

$$\Omega_n = \omega_n^2 a / g \quad , \text{ nth resonant frequency parameter}$$

### Subscripts

F related to surface F

i, j, k related to  $\rho_i, \rho_j, \rho_k$  , respectively

I related to integration variables

P related to the point  $P(r, \psi, \theta)$  or  $P(\rho)$

Q related to the point  $Q(r, \psi, \theta)$  or  $Q(\rho_Q)$

R related to surface R

# APPENDIX I. ANALYTIC EXPRESSION FOR $H_0(p, p')$

The Green's function of the second kind for a whole sphere (Ref. 16) is

$$G_0(P, P') = \frac{1}{4\pi} \left\{ \frac{1}{R} + \frac{a}{rR'} + \frac{1}{a} \ln \frac{2a^2}{a^2 - rr' \cos \gamma + rR'} \right\} \quad [I-1]$$

where

$$R = \sqrt{r^2 + r'^2 - 2rr' \cos \gamma} \quad , \quad R' = \sqrt{r'^2 + \frac{a^4}{r^2} - \frac{2r'}{r} a^2 \cos \gamma} \quad [I-2]$$

$$\cos \gamma = \cos \psi \cos \psi' + \sin \psi \sin \psi' \cos(\theta - \theta') \quad [I-3]$$

When P and P' both on F,

$$G_0(P, P') = \frac{1}{4\pi} \left\{ \frac{1}{\sqrt{p^2 + p'^2 - 2pp' \cos(\theta - \theta')}} + \frac{a}{\sqrt{p^2 p'^2 + (p'^2 + p^2)z_F^2 - 2pp'a^2 \cos(\theta - \theta') + b^4}} + \frac{1}{a} \ln \frac{2a^2}{a^2 - z_F^2 - pp' \cos(\theta - \theta') + \sqrt{p^2 p'^2 + (p'^2 + p^2)z_F^2 - 2pp'a^2 \cos(\theta - \theta') + b^4}} \right\} \quad [I-4]$$

Using Equation [I-4], with  $\sigma = \theta - \theta'$

$$\begin{aligned} H_0(\bar{P}, \bar{P}') &= \frac{1}{\cos \theta} \int_0^{2\pi} G_0(P, P') \cos \theta' d\theta' = \int_0^{2\pi} G_0(r, r'; \cos \sigma; \psi, \psi') \cos \sigma d\sigma \\ &= H_{01} + H_{02} + H_{03} \end{aligned} \quad [I-5]$$

Making use of a new variable  $\beta = \frac{\pi}{2} - \frac{\sigma}{2}$  and a new parameter  $q_1 = \frac{2\sqrt{pp'}}{p+p'}$

$$\begin{aligned} H_{01} &= \frac{1}{4\pi} \int_0^{2\pi} \frac{\cos \sigma d\sigma}{\sqrt{p^2 + p'^2 - 2pp' \cos(\theta - \theta')}} \\ &= \frac{1}{2\pi \sqrt{pp'}} \left\{ \left( \frac{2}{q_1} - q_1 \right) K(q_1) - \frac{2}{q_1} E(q_1) \right\} \end{aligned} \quad [I-6]$$

where

$$K(q) = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1-q^2 \sin^2 \beta}} d\beta$$

$$E(q) = \int_0^{\frac{\pi}{2}} \sqrt{1-q^2 \sin^2 \beta} d\beta$$

This can be evaluated more accurately by

$$H_{01} = \frac{1}{2\pi \sqrt{\rho\rho'}} q_1 [-K(q_1) - 2D(q_1)] \quad [I-6a]$$

where  $D(q_1)$  is DKEF (3) given by Appendix III.

By taking limiting process,

$$H_{01}(0, \rho') = H_{01}(\rho, 0) = 0 \quad [I-6b]$$

Similarly

$$\begin{aligned} H_{02} &= \frac{1}{4\pi} \int_0^{2\pi} \frac{a \cos \sigma d\sigma}{\sqrt{(\rho\rho' - b^2)^2 + 2\rho\rho'a^2 + z_F^2(\rho - \rho')^2 - 2\rho\rho'a^2 \cos \sigma}} \\ &= \frac{1}{2\pi \sqrt{\rho\rho'}} \left\{ \left( \frac{2}{q_2} - q_2 \right) K(q_2) - \frac{2}{q_2} E(q_2) \right\} \end{aligned} \quad [I-7]$$

where

$$q_2 = \frac{2a \sqrt{\rho\rho'}}{\sqrt{(\rho\rho' - b^2)^2 + z_F^2(\rho - \rho')^2 + 4\rho\rho'a^2}} \leq 1$$

Also, this can be evaluated more accurately by

$$H_{02} = \frac{q_2}{2\pi \sqrt{\rho\rho'}} \left\{ -K(q_2) - 2D(q_2) \right\} \quad [I-7a]$$



When  $\rho$  or  $\rho'$  is zero

$$H_{03}(0, \rho') = H_{03}(\rho, 0) = 0$$

[I-7b]

From integration by parts,

$$\begin{aligned} H_{03}(\rho, \rho') &= \frac{1}{4\pi} \int_0^{2\pi} \frac{1}{a} \ln \left[ \frac{2a^2}{a^2 - z_F^2 - \rho\rho' \cos \sigma + \sqrt{(\rho\rho' - b^2)^2 + 2\rho\rho'a^2(1 - \cos \sigma) + z_F^2(\rho - \rho')^2}} \right] \cos \sigma \, d\sigma \\ &= \frac{1}{2\pi a} \int_0^\pi \frac{[\rho\rho' \sqrt{(\rho\rho' - b^2)^2 + 2\rho\rho'a^2(1 - \cos \sigma) + z_F^2(\rho - \rho')^2} + \rho\rho'a^2] \sin^2 \sigma \, d\sigma}{\sqrt{(\rho\rho' - b^2)^2 + 2\rho\rho'a^2(1 - \cos \sigma) + z_F^2(\rho - \rho')^2} [b^2 - \rho\rho' \cos \sigma + \sqrt{(\rho\rho' - b^2)^2 + 2\rho\rho'a^2(1 - \cos \sigma) + z_F^2(\rho - \rho')^2}]} \\ &= \frac{1}{2\pi a} \int_0^\pi \left\{ [\rho\rho' \sqrt{(\rho\rho' - b^2)^2 + 2\rho\rho'a^2(1 - \cos \sigma) + z_F^2(\rho - \rho')^2} + \rho\rho'a^2] \cdot \right. \\ &\quad \cdot \left. \frac{[\sqrt{(\rho\rho' - b^2)^2 + 2\rho\rho'a^2(1 - \cos \sigma) + z_F^2(\rho - \rho')^2} - b^2 + \rho\rho' \cos \sigma] \sin^2 \sigma}{(-\rho^2\rho'^2)(\cos \sigma - \lambda_1)(\cos \sigma - \lambda_2) \sqrt{\rho^2\rho'^2 + b^4 + z_F^2(\rho + \rho')^2 - 2\rho\rho'a^2 \cos \sigma}} \right\} d\sigma \\ &= \frac{\rho\rho'}{2\pi a} \int_0^\pi \left\{ \frac{[\rho^2\rho'^2 + b^4 + z_F^2(\rho + \rho')^2 - 2\rho\rho'a^2 \cos \sigma] - a^2b^2 + a^2\rho\rho' \cos \sigma}{\sqrt{\rho^2\rho'^2 + b^4 + z_F^2(\rho + \rho')^2 - 2\rho\rho'a^2 \cos \sigma} (-\rho^2\rho'^2)(\cos \sigma - \lambda_1)(\cos \sigma - \lambda_2)} \right\} \sin^2 \sigma \, d\sigma + \\ &\quad + \frac{\rho\rho'}{2\pi a} \int_0^\pi \frac{a^2 - b^2 + \rho\rho' \cos \sigma}{-\rho^2\rho'^2(\cos \sigma - \lambda_1)(\cos \sigma - \lambda_2)} d\sigma \\ &= I_1 + I_2 + I_3 + I_4 \end{aligned}$$

[I-8]

where

$$\begin{aligned}
 \frac{\sin^2 \sigma}{(\cos \sigma - \lambda_1)(\cos \sigma - \lambda_2)} &= \frac{1}{(\cos \sigma - \lambda_1)(\cos \sigma - \lambda_2)} - 1 + \frac{-(\lambda_1 + \lambda_2) \cos \sigma + \lambda_1 \lambda_2}{(\cos \sigma - \lambda_1)(\cos \sigma - \lambda_2)} \\
 I_1 &= \frac{\rho \rho'}{2\pi a} \int_0^\pi \frac{\frac{1}{2} [\rho^2 \rho'^2 + b^4 + z_F^2 (\rho^2 + \rho'^2) - 2\rho \rho' a^2 \cos \sigma] + \frac{1}{2} [\rho^2 \rho'^2 + b^4 + z_F^2 (\rho^2 + \rho'^2) - 2a^2 b^2]}{(-\rho^2 \rho'^2) \sqrt{\rho^2 \rho'^2 + b^4 + z_F^2 (\rho + \rho')^2 - 2\rho \rho' a^2 \cos \sigma}} [-1] d\sigma \\
 &= \frac{\rho \rho'}{2\pi a} \left[ -\frac{1}{2} \int_0^{\frac{\pi}{2}} 2 \sqrt{1 - q_2^2 \sin^2 \beta} \cdot \frac{2\sqrt{\rho \rho'} a}{q_2} \cdot \left( \frac{-1}{\rho^2 \rho'^2} \right) d\beta + \right. \\
 &\quad \left. + \frac{\rho \rho'}{2\pi a} \left[ \frac{1}{\rho^2 \rho'^2} \int_0^{\frac{\pi}{2}} 2 \cdot \frac{\frac{1}{2} [\rho^2 \rho'^2 + b^4 + z_F^2 (\rho^2 + \rho'^2) - 2a^2 b^2]}{\sqrt{1 - q_2^2 \sin^2 \beta} (2a \sqrt{\rho \rho'} / q_2)} d\beta \right] \right. \\
 &= \frac{1}{2\pi a} \left\{ \frac{1}{\rho \rho'} \left( \frac{2\sqrt{\rho \rho'} a}{q_2} \right) E(q_2) \right\} + \\
 &\quad + \frac{1}{2\pi a} \left\{ \frac{1}{\rho \rho'} \left( \frac{q_2}{2a \sqrt{\rho \rho'}} \right) [\rho^2 \rho'^2 + b^4 + z_F^2 (\rho^2 + \rho'^2) - 2a^2 b^2] K(q_2) \right\}
 \end{aligned}$$

[I-8a]

By using partial fractions, the variable  $\beta$  and the definition of  $\Pi(\alpha^2, q_2)$  (c.f. Ref. 9), one finds.

$$\begin{aligned}
I_2 &= \frac{\rho\rho'}{2\pi a} \int_0^\pi \frac{[\rho^2\rho'^2 + b^4 + z_F^2(\rho+\rho')^2 - a^2b^2] - \rho\rho'a^2\cos\sigma}{(-\rho^2\rho'^2)\sqrt{\rho^2\rho'^2 + b^4 + z_F^2(\rho+\rho')^2 - 2\rho\rho'a^2\cos\sigma}} \left[ \frac{1}{(\cos\sigma - \lambda_1)(\cos\sigma - \lambda_2)} + \frac{-(\lambda_1 + \lambda_2)\cos\sigma + \lambda_1\lambda_2}{(\cos\sigma - \lambda_1)(\cos\sigma - \lambda_2)} \right] d\sigma \\
&= -\frac{1}{2\pi a\rho\rho'} \int_0^\pi \frac{[\rho^2\rho'^2 + b^4 + z_F^2(\rho+\rho')^2 - a^2b^2] - \rho\rho'a^2\cos\sigma}{\sqrt{\rho^2\rho'^2 + b^4 + z_F^2(\rho+\rho')^2 - 2\rho\rho'a^2\cos\sigma}} \left\{ \frac{1+\lambda_1\lambda_2}{\lambda_1-\lambda_2} \left( \frac{1}{\cos\sigma - \lambda_1} - \frac{1}{\cos\sigma - \lambda_2} \right) - \right. \\
&\quad \left. - \frac{\lambda_1+\lambda_2}{\lambda_1-\lambda_2} \left( \frac{\lambda_1}{\cos\sigma - \lambda_1} - \frac{\lambda_2}{\cos\sigma - \lambda_2} \right) \right\} d\sigma + \\
&\quad + \frac{1}{2\pi a} \cdot a^2 \int_0^\pi \frac{1}{\sqrt{\rho^2\rho'^2 + b^4 + z_F^2(\rho+\rho')^2 - 2\rho\rho'a^2\cos\sigma}} \left\{ \left( \frac{\lambda_1}{\cos\sigma - \lambda_1} - \frac{\lambda_2}{\cos\sigma - \lambda_2} \right) \frac{1+\lambda_1\lambda_2}{\lambda_1-\lambda_2} - \right. \\
&\quad \left. - \frac{\lambda_1+\lambda_2}{\lambda_1-\lambda_2} \left[ (\lambda_1 - \lambda_2) + \frac{\lambda_1^2}{\cos\sigma - \lambda_1} - \frac{\lambda_2^2}{\cos\sigma - \lambda_2} \right] \right\} d\sigma \\
&= -\frac{1}{2\pi a\rho\rho'} [\rho^2\rho'^2 + b^4 + z_F^2(\rho^2 + \rho'^2) - a^2b^2] \left[ \frac{q_2}{a\sqrt{\rho\rho'}} \right] \left\{ \frac{\lambda_1-1}{\lambda_1-\lambda_2} \Pi\left(\frac{2}{1+\lambda_1}, q_2\right) + \right. \\
&\quad + \frac{1-\lambda_2}{\lambda_1-\lambda_2} \Pi\left(\frac{-2}{-\lambda_2-1}, q_2\right) \left. \right\} + \frac{a^2}{2\pi a} \left[ \frac{q_2}{a\sqrt{\rho\rho'}} \right] \left\{ -(\lambda_1+\lambda_2) K(q_2) + \right. \\
&\quad + \frac{(\lambda_1-1)\lambda_1}{\lambda_1-\lambda_2} \Pi\left(\frac{2}{1+\lambda_1}, q_2\right) + \frac{(1-\lambda_2)\lambda_2}{\lambda_1-\lambda_2} \Pi\left(\frac{-2}{-\lambda_2-1}, q_2\right) \left. \right\}
\end{aligned}$$

[I-8b]

where the well-known complete elliptic integral of the third kind is

defined by

$$\Pi(\alpha^2, q) = \int_0^{\frac{\pi}{2}} \frac{d\beta}{(1 - \alpha^2 \sin^2 \beta) \sqrt{1 - q^2 \sin^2 \beta}} = \int_0^{K(q)} \frac{du}{(1 - \alpha^2 \operatorname{sn} u)}$$

[I-9]

$$\begin{aligned}
I_3 &= \frac{\rho\rho'}{2\pi a} \left(-\frac{1}{\rho^2\rho'^2}\right) \int_0^\pi -(a^2-b^2+\rho\rho'\cos\sigma) d\sigma \\
&= \frac{1}{2\pi a} \left(\frac{1}{\rho\rho'}\right) [(a^2-b^2)\pi] \quad [I-8c]
\end{aligned}$$

$$\begin{aligned}
I_4 &= -\frac{1}{2\pi a\rho\rho'} \int_0^\pi \frac{1}{\rho^2\rho'^2} \left[ \frac{(a^2-b^2)+\rho\rho'\cos\sigma}{(\cos\sigma-\lambda_1)(\cos\sigma-\lambda_2)} + \frac{(a^2-b^2+\rho\rho'\cos\sigma)(\lambda_1\lambda_2-(\lambda_1+\lambda_2)\cos\sigma)}{(\cos\sigma-\lambda_1)(\cos\sigma-\lambda_2)} \right] d\sigma \\
&= \frac{-1}{2\pi a\rho\rho'} \int_0^\pi \left\{ -(\lambda_1+\lambda_2)\rho\rho' + \frac{1}{(\cos\sigma-\lambda_1)(\cos\sigma-\lambda_2)} \left[ \rho\rho'(1+\lambda_1\lambda_2-(\lambda_1+\lambda_2)^2) - (\lambda_1+\lambda_2)(a^2-b^2) \right] + \right. \\
&\quad \left. + (\lambda_1+\lambda_2)\lambda_1\lambda_2\rho\rho' + (1+\lambda_1\lambda_2)(a^2-b^2) \right\} d\sigma \\
&= \frac{1}{2\pi a} \left\{ (\lambda_1+\lambda_2)\pi - \frac{\pi}{\rho\rho'} \sqrt{\lambda_1^2-1} [\rho\rho'\lambda_1+(a^2-b^2)] - \frac{\pi}{\rho\rho'} \sqrt{\lambda_2^2-1} [\rho\rho'\lambda_2+(a^2-b^2)] \right\} \quad [I-8d]
\end{aligned}$$

where

$$a^2-b^2 = z_F^2$$

Hence

$$\begin{aligned}
H_{0,3}(\rho, \rho') &= \frac{1}{2\pi a} \left\{ \left[ \frac{1}{\rho\rho'} \left( \frac{2\sqrt{\rho\rho'}a}{g_2} \right) E(g_2) \right] + \frac{1}{\rho\rho'} \left( \frac{g_2}{2a\sqrt{\rho\rho'}} \right) [\rho^2\rho'^2+b^4+z_F^2(\rho^2+\rho'^2)-2a^2b^2] K(g_2) - \right. \\
&\quad \left. - 2 \left( \frac{g_2}{2a\sqrt{\rho\rho'}} \right) \left[ \frac{1}{\rho\rho'} (\rho^2\rho'^2+b^4+z_F^2(\rho^2+\rho'^2)-a^2b^2) \right] \left[ \frac{\lambda_1-1}{\lambda_1-\lambda_2} \Pi\left(\frac{2}{1+\lambda_1}, g_2\right) + \frac{1-\lambda_2}{\lambda_1-\lambda_2} \Pi\left(\frac{-2}{-\lambda_2-1}, g_2\right) \right] + \right. \\
&\quad \left. + 2a^2 \left( \frac{g_2}{2a\sqrt{\rho\rho'}} \right) \left[ -(\lambda_1+\lambda_2)K(g_2) + \frac{(\lambda_1-1)\lambda_1}{\lambda_1-\lambda_2} \Pi\left(\frac{2}{1+\lambda_1}, g_2\right) + \frac{(1-\lambda_2)\lambda_2}{\lambda_1-\lambda_2} \Pi\left(\frac{-2}{-\lambda_2-1}, g_2\right) + \right. \right. \\
&\quad \left. \left. + \frac{1}{\rho\rho'} (a^2-b^2)\pi + (\lambda_1+\lambda_2)\pi - \frac{\pi}{\rho\rho'} \sqrt{\lambda_1^2-1} [\rho\rho'\lambda_1+z_F^2] - \frac{\pi}{\rho\rho'} \sqrt{\lambda_2^2-1} [\rho\rho'\lambda_2+z_F^2] \right] \right\} \quad [I-8']
\end{aligned}$$

When  $\rho$  or  $\rho'$  is zero,

$$H_{03}(0, \rho') = H_{03}(\rho, 0) = 0 \quad [I-8'']$$

It was found that there is a precision problem in Equation [I-8'] when  $\rho$  and  $\rho'$  are near zero. This might be anticipated as there is a very small denominator proportional to  $(\rho\rho')^{3/2}$  and the result is expected to be small in view of Equation [I-8'']. After somewhat laborious manipulations with Equation [I-8'] to resolve the precision problem,  $H_{03}$  is obtained in the following form (with  $b = 1$ ):

$$\begin{aligned} H_{03} = & \frac{1}{2\pi a} \left\{ \frac{1}{\rho\rho'Q_s} g_2 D(g_2) + \frac{1}{\rho\rho'R_{th}} (R_{th}^2 - 1) P(g_2) - \frac{1}{\rho\rho'} Q_s [\rho^2\rho'^2 + z_F^2(\rho^2 + \rho'^2)] P(g_2) - \right. \\ & - \frac{2Q_s}{\rho\rho'} [\rho^2\rho'^2 + z_F^2(\rho^2 + \rho'^2) + (b^2 - a^2b^2)] \left[ \frac{\lambda_1 - 1}{\lambda_1 - \lambda_2} \Pi\left(\frac{2}{1+\lambda_1}, g_2\right) + S_e \right] + \\ & \left. + 2a^2Q_s \left[ \frac{(1-\lambda_1)(1+\lambda_1)}{\lambda_1 - \lambda_2} P(g_2) + \frac{(\lambda_1 - 1)\lambda_1}{\lambda_1 - \lambda_2} \Pi\left(\frac{2}{1+\lambda_1}, g_2\right) + \frac{(1-\lambda_2^2)}{\lambda_1 - \lambda_2} \Pi_N + R_{H3} \right] \right\} \end{aligned} \quad [I-10]$$

where

$$Q_s = \frac{g_2}{2a\sqrt{\rho\rho'}} \quad [I-10a]$$

$$R_{th} = \sqrt{(\rho\rho' - b^2)^2 + z_F^2(\rho - \rho')^2 + 4\rho\rho'a^2} \quad [I-10b]$$

$$S_e = - \frac{(\lambda_1 - 1)}{\lambda_1 - \lambda_2} \left[ \frac{1}{\lambda_2} K(g_2) + P(g_2) \right] + \frac{K(g_2)}{\lambda_2} - \frac{(\lambda_2^2 - 1)}{(\lambda_1 - \lambda_2)\lambda_2} \Pi_N \quad [I-10c]$$

$$R_{H3} = R_1 + R_2 + R_3 + R_4 \quad [I-10d]$$

$$R_1 = \frac{1}{\rho\rho'} \left\{ \left[ \rho^2 \rho'^2 + 2\rho\rho' + z_F^2 (\rho + \rho')^2 \right] \frac{1}{R_{Hh}} \cdot \frac{\pi}{2} + Q_s \left[ \rho^2 \rho'^2 + z_F^2 (\rho^2 + \rho'^2) \right] \frac{\pi}{2} \right. \\ \left. - 2Q_s \left[ \rho^2 \rho'^2 + z_F^2 (\rho^2 + \rho'^2) \right] \frac{(1-\lambda_2^2)}{(\lambda_1 - \lambda_2)(1+\lambda_2)} \cdot \frac{\pi}{2} - 2Q_s \frac{(1-\lambda_1)}{\lambda_1 - \lambda_2} (b^4 - a^2 b^2) \frac{\pi}{2} \right\} \quad [I-10e]$$

$$R_2 = 2a^2 Q_s \frac{(1-\lambda_1)(1+\lambda_1)}{\lambda_1 - \lambda_2} \cdot \frac{\pi}{2} \quad [I-10f]$$

$$R_3 = \frac{\pi}{2\rho\rho'} \left\{ \frac{z_F^2 (\rho^2 + \rho'^2) + \rho^2 \rho'^2}{R_{Hh} + z_F^2} + \frac{\rho^2 + \rho'^2}{2} \right\} \cdot \left\{ 1 + \sqrt{1 + \frac{1}{2z_F^2} \left( \frac{z_F^2 (\rho^2 + \rho'^2) + \rho^2 \rho'^2}{R_{Hh} + z_F^2} + \frac{\rho^2 + \rho'^2}{2} \right)} \right\}^{-1} \quad [I-10g]$$

$$R_{tq} = \sqrt{z_F^4 + z_F^2 (\rho^2 + \rho'^2) + \rho^2 \rho'^2} \quad [I-10h]$$

$$R_4 = -\frac{\pi}{2} \sqrt{(\lambda_1 - 1)(\lambda_1 + 1)} \quad [I-10i]$$

$$\lambda_1 > 1, \quad \lambda_2 < 0 \quad [I-10j]$$

It must be noted that  $(\lambda_1 - 1)$ ,  $(\lambda_2^2 - 1)$ ,  $(R_{Hh}^2 - 1)$  are evaluated not by direct subtractions, but by accurate formulas.

$$(R_{Hh}^2 - 1) = \rho^2 \rho'^2 + 2\rho\rho' + z_F^2 (\rho + \rho')^2 \quad [I-10k]$$

$$(\lambda_1 - 1) = R_{ti}^2 / (1 + \lambda_1) \quad [I-10l]$$

$$R_{ti} = \frac{|\rho - \rho'|}{\rho \rho'} \cdot \frac{|z_F| \sqrt{2z_F^2 + 2R_{ta} + (\rho^2 + \rho'^2)}}{z_F^2 + \rho \rho' + R_{ta}} \quad [I-10m]$$

$$(\lambda_2^2 - 1) = \left[ |z_F| \sqrt{2z_F^2 + 2R_{ta} + (\rho^2 + \rho'^2)} \right]^2 / (\rho^2 \rho'^2) \quad [I-10n]$$

$(\rho - \rho')$  can be calculated without loss of significant figure. It is also noted that for small  $z_F$ , one should replace  $(b^4 - b^2 a^2) = b^2(b^2 - a^2)$  by  $-z_F^2 b$  or  $(1 - a^2)$  by  $-z_F^2$  with  $b = 1$ .

It is recalled that Budiansky's technique of differentiation under integral sign does not seem to lead to simple results, due to the presence of non-zero  $z_F$ , the relative depth measured from the center of the spherical tank.

## APPENDIX II. ANALYTIC EXPRESSION FOR $\mathcal{F}(\rho, \rho')$

The outer normal derivative of  $G_0(\rho, \rho')$  on the free surface is

$$\begin{aligned} \frac{\partial G_0}{\partial z'} = & -\frac{1}{4\pi} \left\{ \frac{1}{R^3} (z' - z) + \frac{1}{R^3} \left( \frac{a}{r} z' - \frac{a^3}{r^3} z \right) + \frac{1}{a r^2 r'^2 (\cos^2 \gamma - 1)} [z r r' \cos \gamma - r^2 z'] + \right. \\ & \left. + \frac{1}{a R' r^2 r'^2 (\cos^2 \gamma - 1)} [(a^2 r z' + r r'^2 z) - (r^2 r' z' + a^2 r' z) \cos \gamma] \right\} \end{aligned} \quad [\text{II-1}]$$

When both P and P' on F,

$$\begin{aligned} \left. \frac{\partial G_0}{\partial z'} \right|_{z=z'=z_F} = & -\frac{1}{4\pi} \left\{ \frac{z_F}{R^3} \left( \frac{a}{r} - \frac{a^3}{r^3} \right) + \frac{z_F}{a r^2 r'^2 (\cos^2 \gamma - 1)} [r r' \cos \gamma - r^2] + \right. \\ & \left. + \frac{z_F}{a R' r^2 r'^2 (\cos^2 \gamma - 1)} [(a^2 r + r r'^2) - (r^2 r' + a^2 r') \cos \gamma] \right\} \end{aligned} \quad [\text{II-2}]$$

where

$$r^2 = \rho^2 + z_F^2, \quad r'^2 = \rho'^2 + z_F^2$$

For P, P' both on F,

$$\begin{aligned} \mathcal{F}(\bar{\rho}, \bar{\rho}') &= \int_0^{2\pi} \left. \frac{\partial G_0}{\partial z'} \right|_{z=z'=z_F} \cos \sigma \, d\sigma \\ &= \mathcal{F}_1 + \mathcal{F}_2 + \mathcal{F}_3 \end{aligned} \quad [\text{II-3}]$$

Using the same technique as in Appendix I, one finds

$$\begin{aligned} \mathcal{F}_1(\rho, \rho') &= \int_0^{2\pi} -\frac{1}{4\pi} \cdot \frac{z_F}{R^3} \left( \frac{a}{r} - \frac{a^3}{r^3} \right) \cos \sigma \, d\sigma \\ &= \frac{(a^3 - r^2 a) z_F}{\pi} \frac{q_2^3}{(4\rho\rho'a^2)^{3/2}} \left\{ \left( \frac{2}{q_2^2} - 1 \right) \frac{1}{1 - q_2^2} E(q_2) - \frac{2}{q_2^2} K(q_2) \right\} \end{aligned} \quad [\text{II-4}]$$

where  $a^3 - r^2 a = a(b^2 - \rho^2)$



It is noted that a special case of the reduction formula in Reference 13 can be used to evaluate the following integral, which occurred in  $\mathcal{F}(\rho, \rho')$

$$\int_0^{\frac{\pi}{2}} \frac{d\beta}{[1 - g_2^2 \sin^2 \beta]^{3/2}} = \frac{1}{1 - g_2^2} E(g_2)$$

Or, use

$$\frac{K^2 \sin^2 \phi \cos \phi}{\sqrt{1 - K^2 \sin^2 \phi}} = \int_0^\phi \frac{-(1 - K^2) d\phi}{(1 - K^2 \sin^2 \phi)^{3/2}} + \int_0^\phi \sqrt{1 - K^2 \sin^2 \phi} d\phi$$

which can be checked easily by differentiation.

When  $\rho$  or  $\rho'$  is zero,

$$\mathcal{F}(0, \rho) = \mathcal{F}(\rho, 0) = 0 \quad [\text{II-4a}]$$

To increase accuracy in numerical evaluation, Equation [II-4] is replaced by (with  $b = 1$ )

$$\mathcal{F}(\rho, \rho') = \left[ \frac{E(g_2)}{1 - g_2^2} + 2D(g_2) \right] \frac{Q_5^3 \mathcal{F}_F a}{\pi} (b + \rho)(b - \rho) \quad [\text{II-4b}]$$

$b - \rho$ ,  $\rho - \rho'$  and  $b^2 - \rho\rho'$  can be evaluated accurately for known discrete values of  $\rho, \rho'$  and  $(1 - g_2^2)$  is evaluated by

$$g_2' = \sqrt{(1 - g_2^2)} = \sqrt{\frac{(b^2 - \rho\rho')^2 + \mathcal{F}_F^2 (\rho - \rho')^2}{4\rho\rho'a^2 + (b^2 - \rho\rho')^2 + \mathcal{F}_F^2 (\rho - \rho')^2}} \quad [\text{II-4c}]$$

Next,  $\mathcal{F}_2$  will be expressed in terms of elementary functions.

$$\begin{aligned}
\mathcal{J}_2(\rho, \rho') &= \frac{-1}{4\pi a} \int_0^{2\pi} \frac{z_F}{r^2 r'^2 (\cos^2 \theta - 1)} [r r' \cos \theta - r^2] \cos \theta \, d\theta \\
&= \frac{-z_F}{2\pi a} \int_0^\pi \frac{z_F^2 + \rho \rho' \cos \sigma - (z_F^2 + \rho^2)}{(z_F^2 + \rho \rho' \cos \sigma)^2 - (z_F^2 + \rho^2)(z_F^2 + \rho'^2)} \cos \sigma \, d\sigma \\
&= \frac{-z_F}{2\pi a} \left\{ \frac{\pi}{\rho \rho'} + \int_0^\pi \frac{(-2z_F^2 - \rho^2) \cos \sigma + [z_F^2(\rho^2 + \rho'^2) + \rho^2 \rho'^2]/(\rho \rho')}{\rho^2 \rho'^2 [\cos \sigma - \lambda_1][\cos \sigma - \lambda_2]} d\sigma \right\} \\
&= \frac{-z_F}{2\rho \rho' a} - \frac{z_F}{2a\rho^2 \rho'^2 (\lambda_1 - \lambda_2)} \left[ \frac{z_F^2(\rho^2 + \rho'^2) + \rho^2 \rho'^2}{(-\rho \rho')} + \lambda_1 (2z_F^2 + \rho^2) \right] \frac{1}{\sqrt{\lambda_1^2 - 1}} - \\
&\quad - \frac{z_F}{2\rho^2 \rho'^2 a (\lambda_1 - \lambda_2)} \left[ \frac{z_F^2(\rho^2 + \rho'^2) + \rho^2 \rho'^2}{(-\rho \rho')} + \lambda_2 (2z_F^2 + \rho^2) \right] \frac{1}{\sqrt{\lambda_2^2 - 1}} \\
&= \frac{-z_F}{2\rho \rho' a} \left\{ 1 + \frac{1}{2\sqrt{1 - (\frac{1}{\lambda_1})^2}} \left[ \frac{(z_F^2 + \rho^2)\sqrt{z_F^4 + z_F^2(\rho^2 + \rho'^2) + \rho^2 \rho'^2}}{z_F^4 + z_F^2(\rho^2 + \rho'^2) + \rho^2 \rho'^2} - 1 \right] + \right. \\
&\quad \left. + \frac{1}{2\sqrt{1 - (\frac{1}{\lambda_2})^2}} \left[ -\frac{(z_F^2 + \rho^2)\sqrt{z_F^4 + (\rho^2 + \rho'^2)z_F^2 + \rho^2 \rho'^2}}{z_F^4 + z_F^2(\rho^2 + \rho'^2) + \rho^2 \rho'^2} - 1 \right] \right\}
\end{aligned}$$

[II-5]

Some further manipulation is required to avoid precision problems

for  $\rho, \rho'$  small or  $z_F$  small. One finds

$$\begin{aligned}
\mathcal{J}_2 &= \frac{-z_F}{2\rho \rho' a} \left\{ \frac{1}{2} \left[ \frac{z \rho^2 \rho'^2 + (z_F^2 + R_{ta})[(\rho^2 + \rho'^2) - |\rho^2 - \rho'^2|]}{|z_F| R_{tj}} \right] \right. \\
&\quad \left. + \frac{1}{R_{ta}} \cdot \frac{z_F^2 + \rho^2}{z_F^2 + \rho^2 + R_{ta}} \cdot \frac{(\rho - \rho')}{|\rho - \rho'|} - \frac{\rho^2 \rho'^2}{|z_F| R_{tj} [|z_F| R_{tj} + z_F^2 + R_{ta}]} \right\}
\end{aligned}$$

[II-5a]

$$R_{tj} = \sqrt{2z_F^2 + 2R_{ta} + \rho^2 + \rho'^2}$$

[II-5b]

When  $\rho$  or  $\rho' \rightarrow 0$

$$\mathcal{F}_2(0, \rho') = \mathcal{F}_2(\rho, 0) = 0$$

[II-5c]

Finally,  $\mathcal{F}_3$  will be expressed in closed form as follows:

$$\begin{aligned} \mathcal{F}_3(\rho, \rho') &= -\frac{z_F}{4\pi} \int_0^{2\pi} \frac{1}{a r R' r^2 r'^2 (\cos^2 \gamma - 1)} [(a^2 r^2 + r^2 r'^2) - (r^2 + a^2) r r' \cos \gamma] \cos \sigma d\sigma \\ &= -\frac{z_F}{2\pi a} \int_0^\pi \frac{[(a^2 r^2 + r^2 r'^2) - (r^2 + a^2)(z_F^2 + \rho \rho' \cos \sigma)]}{\sqrt{(\rho \rho' - b^2)^2 + 2\rho \rho' a^2 + z_F^2(\rho - \rho')^2 - 2\rho \rho' a^2 \cos \sigma}} \\ &\quad \cdot \frac{\cos \sigma d\sigma}{[-z_F^2(\rho^2 + \rho'^2 - 2\rho \rho' \cos \sigma) - \rho^2 \rho'^2(1 - \cos^2 \sigma)]} \\ &= \frac{-z_F}{\pi a} \int_0^{\frac{\pi}{2}} \frac{g_2}{2\sqrt{\rho \rho'}} \left\{ \frac{1}{\sqrt{1 - g_2^2 \sin^2 \beta}} \right\} \cdot \left\{ \left[ \frac{-(r^2 + a^2)}{\rho \rho'} \right] + \frac{1}{\rho^2 \rho'^2} \left( \frac{1}{\lambda_1 - \lambda_2} \right) \left( \frac{1}{\cos \sigma - \lambda_1} \right) \cdot \right. \\ &\quad \cdot [\lambda_1 (r^2 + a^2) z_F^2 + r^2 (a^2 + r'^2)] - \frac{r^2 + a^2}{\rho \rho'} (z_F^2(\rho^2 + \rho'^2) + \rho^2 \rho'^2)] + \\ &\quad \left. + \frac{1}{\rho^2 \rho'^2} \left( \frac{1}{\lambda_1 - \lambda_2} \right) \left( \frac{-1}{\cos \sigma - \lambda_2} \right) [\lambda_2 (r^2 + a^2) z_F^2 + r^2 (a^2 + r'^2)] - \frac{r^2 + a^2}{\rho \rho'} (z_F^2(\rho^2 + \rho'^2) + \rho^2 \rho'^2)] \right\} d\beta \\ &= \frac{-z_F g_2}{2\pi a^2 (\rho \rho')^{3/2}} \left\{ -(\rho^2 + z_F^2 + a^2) K(g_2) + \frac{1}{2\sqrt{z_F^4 + z_F^2(\rho^2 + \rho'^2) + \rho^2 \rho'^2}} \cdot \right. \\ &\quad \cdot \frac{(-\rho \rho')}{\rho \rho' - z_F^2 + \sqrt{z_F^4 + z_F^2(\rho^2 + \rho'^2) + \rho^2 \rho'^2}} \cdot [\lambda_1 ((z_F^2 + \rho^2 + a^2) z_F^2 + (z_F^2 + \rho^2)(a^2 + z_F^2 + \rho'^2)) - \\ &\quad - (z_F^2 + a^2 + \rho^2) \frac{z_F^2(\rho^2 + \rho'^2) + \rho^2 \rho'^2}{\rho \rho'}] \cdot \Pi\left(\frac{2}{1 + \lambda_1}, g_2\right) + \frac{1}{2\sqrt{z_F^4 + z_F^2(\rho^2 + \rho'^2) + \rho^2 \rho'^2}} \cdot \\ &\quad \cdot \frac{(-\rho \rho')}{z_F^2 - \rho \rho' + \sqrt{z_F^4 + z_F^2(\rho^2 + \rho'^2) + \rho^2 \rho'^2}} \cdot [\lambda_2 ((z_F^2 + a^2 + \rho^2) z_F^2 + (z_F^2 + \rho^2)(a^2 + z_F^2 + \rho'^2)) - \\ &\quad - (z_F^2 + \rho^2 + a^2) \frac{z_F^2(\rho^2 + \rho'^2) + \rho^2 \rho'^2}{\rho \rho'}] \cdot \Pi\left(\frac{-2}{-\lambda_2 - 1}, g_2\right) \left. \right\} \end{aligned}$$

$$\begin{aligned}
&= \frac{z_F q_2}{2\pi a^2 (\rho \rho')^{3/2}} \left\{ (z_F^2 + \rho^2 + a^2) K(q_2) + \frac{1}{2} \left[ 1 - \frac{(\rho \rho' - z_F^2) \sqrt{z_F^4 + z_F^2(\rho^2 + \rho'^2) + \rho^2 \rho'^2}}{z_F^4 + z_F^2(\rho^2 + \rho'^2) + \rho^2 \rho'^2} \right] \right. \\
&\quad \cdot \frac{1}{z_F^2(\rho + \rho')^2} \left[ \left( -z_F^2 + \sqrt{z_F^4 + z_F^2(\rho^2 + \rho'^2) + \rho^2 \rho'^2} \right) \left( 2z_F^4 + z_F^2(2a^2 + 2\rho^2 + \rho'^2) + \rho^2(a^2 + \rho'^2) \right) - \right. \\
&\quad \left. \left. - (z_F^2 + a^2 + \rho^2) \left( z_F^2(\rho^2 + \rho'^2) + \rho^2 \rho'^2 \right) \right] \Pi\left(\frac{2}{1+\lambda_1}, q_2\right) - \right. \\
&\quad \left. - \frac{1}{2} \left[ 1 + \frac{(\rho \rho' - z_F^2) \sqrt{z_F^4 + z_F^2(\rho^2 + \rho'^2) + \rho^2 \rho'^2}}{z_F^4 + z_F^2(\rho^2 + \rho'^2) + \rho^2 \rho'^2} \right] \cdot \frac{1}{z_F^2(\rho + \rho')^2} \right. \\
&\quad \cdot \left[ \left( z_F^2 + \sqrt{z_F^4 + z_F^2(\rho^2 + \rho'^2) + \rho^2 \rho'^2} \right) \left( 2z_F^4 + z_F^2(2a^2 + 2\rho^2 + \rho'^2) + \rho^2(a^2 + \rho'^2) \right) + \right. \\
&\quad \left. \left. + (z_F^2 + \rho^2 + a^2) \left( z_F^2(\rho^2 + \rho'^2) + \rho^2 \rho'^2 \right) \right] \Pi\left(\frac{-2}{-\lambda_2 - 1}, q_2\right) \right\} \quad [\text{II-6}]
\end{aligned}$$

There is a serious precision problem for  $\rho, \rho'$  small in Equation [II-6]. After manipulations, the precision problem is resolved by employing the following equivalent form.

$$\mathcal{F}_3 = \frac{z_F q_2}{2\pi a^2 (\rho \rho')^{3/2}} \{ F_{31} + F_{32} \} \quad [\text{II-7}]$$

where

$$F_{31} = - \left\{ (z_F^2 + \rho^2 + a^2) \Pi_N + \frac{1}{2} [B_2 + 2(z_F^2 + \rho^2 + a^2) B_1 + B_1 B_2] (K(q_2) + \Pi_N) \right\} \quad [\text{II-7a}]$$

$$B_1 = (\rho^2 - \rho'^2) \frac{(z_F^2 + \rho^2)}{R_{ta} (R_{ta} + z_F^2 + \rho^2)} \quad [\text{II-7b}]$$

$$B_2 = (\rho'^2 - \rho^2) \left\{ 1 + \frac{z_F^2 + \rho^2 + \rho'^2}{z_F^2 + R_{ta}} \left[ 1 - \frac{\rho^2}{z_F^2 + \rho^2 + R_{ta}} \right] \right\} \quad [\text{II-7c}]$$

Where the terms in the inner bracket could be replaced by  $\frac{z_F^2 + R_{ta}}{z_F^2 + \rho^2 + R_{ta}}$  for higher precision, which seems unnecessary as the error in  $B_2$  is sufficiently small in the critical range due to the factor  $\rho'^2 - \rho^2$ .

$$F_{32} = C_1^* C_2^* C_3^* \Pi\left(\frac{z}{1+\lambda_1}, q_z\right) \quad [\text{II-7d}]$$

$$\begin{aligned} C_1^* = 0.5 \left\{ \frac{z}{z_F^2 + \rho^2} - (\rho'^2 - \rho^2) \frac{z_F^2}{(\rho^2 + z_F^2) R_{ta} [z_F^2 + \rho^2 + R_{ta}]} + \right. \\ + \frac{\rho^3 (\rho'^2 - \rho^2)}{(\rho + \rho') (\rho^2 + z_F^2) R_{ta} [\rho^2 + z_F^2 + R_{ta}]} \cdot \left[ 1 + \frac{z_F^2 + \rho^2 + \rho'^2}{\rho \rho' + R_{ta}} \right] + \\ + \frac{(\rho - \rho')}{\rho' (z_F^2 + \rho^2) R_{ta}} \cdot \frac{(z_F^2 + \rho^2)^2 (z_F^2 + \rho^2 + \rho'^2) + (z_F^2 + 2\rho^2) \rho^2 \rho'^2}{(z_F^2 + \rho^2) R_{ta} + \rho^3 \rho'} - \\ \left. - \frac{(\rho - \rho')}{\rho' R_{ta} [\rho \rho' + R_{ta}]} \right\} \quad [\text{II-7e}] \end{aligned}$$

where further manipulation may be needed for very small  $z_F$  to avoid precision problems in the domain of small  $\rho$  and  $\rho'$ ,

$$C_2^* = \frac{1}{\rho + \rho'} \quad [\text{II-7 f}]$$

$$C_3^* = -(\rho - \rho')(z_F^2 + \rho^2) \left[ -(2z_F^2 + \rho^2 + a^2) + R_{za} + \frac{(z_F^2 + \rho^2 + a^2)(2z_F^2 + \rho^2)}{R_{za} + (z_F^2 + \rho^2)} \right] \quad [\text{II-7 g}]$$

When  $\rho$  or  $\rho' \rightarrow 0$ ,  $\mathcal{F}_3 \rightarrow 0$ .

It is important to note that whether  $\rho > \rho'$  or  $\rho' > \rho$  the sum of  $\mathcal{F}_1$ ,  $\mathcal{F}_2$  and  $\mathcal{F}_3$  always approaches zero as  $z_F \rightarrow 0$ . Therefore, for a half-full tank  $H = H_0$  which is in agreement with Budiansky's kernel function aside from an apparent factor of two difference mentioned previously.

## APPENDIX III. SUBROUTINES DKEF AND NEFF (WIZ PROGRAM)

$$\text{DKEF} = \text{DKEF} (k, k, 1, 1, 1)$$

$$\text{NEFF} = \text{NEFF} (\beta, \kappa, \kappa', 1, 1)$$

The unity arguments are actually dummies, while the five arguments represent five outputs. For DKEF, the outputs are  $K(k) = \text{DKEF}(1)$ ,  $E(k) = \text{DKEF}(2)$ ,  $(E(k) - K(k))/k^2 = \text{DKEF}(3)$ ,  $K(k) - \pi/2 = \text{DKEF}(4)$ , and the number of iterations =  $\text{DKEF}(5)$ .  $\text{DKEF}(3)$  is not obtained from  $\text{DKEF}(1) - \text{DKEF}(2)$  but is obtained after a significant analytic subtraction in the program. For NEFF the outputs are  $F(\beta, k = \kappa) = \text{NEFF}(1)$ ,  $E(\beta, k = \kappa) = \text{NEFF}(2)$ ,  $(E(\beta, k) - k \sin \beta) = \text{NEFF}(3)$ , the number of iterations for evaluating  $F(\beta, k) = \text{NEFF}(4)$ , the number of iterations for evaluating  $\text{NEFF}(3) = \text{NEFF}(5)$ .  $\text{NEFF}(3)$  is evaluated after a significant analytic subtraction in the program while  $\text{NEFF}(2)$  is simply obtained from  $\text{NEFF}(3) + k \sin \beta$ . Although  $k' = \sqrt{1 - k^2}$  does not appear in the functions sought, it is calculated from a formula without subtraction, as one can easily see significant figures of  $k'$  would be lost if  $k$  is near unity. The basic formulae are all given in Reference 13. For complete elliptic integrals, the iterative method based on geometric and arithmetic means was employed. For incomplete elliptical integrals, the iterative method based on inverse order of transformation was employed in order to construct  $\text{NEFF}(3)$ . The programs are written in "WIZ" language for GE 225 computers, which is analogous to "FORTRAN" for IBM computers, and are given on the following pages:

## WIZ SOURCE PROGRAM

SEQ	LABL	TY	STATEMENT	C	ZE	NZE	PL	MI	ANY
400	DKEF	.	_ARG#DKEF(1)                      \$DKEF(K,KP,1,1,1)						
401			_ARGP#DKEF(2)						
402			_VA#1,VB#ARGP,PI#3.1415926536						
403			_QROD#ARG*ARG/((1&ARGP)*(1&ARGP)),KN#QROD.NO#0						
404			_KK#0.5*_QROD,SUN#-0.5*(1&KK)						
410	*		_KP#VB/VA,SUM#SUN,PN# 2*SQRT.(VB*VA)/(VA&VB)						
411			_KN#KN*KN/((1&PN)*(1&PN)),ROD2#KN,ROD1#_QROD						
412			_QROD#ROD1&ROD2&ROD1*ROD2						
420			_KK#KK*0.5*KN,SUN#SUM-0.5*KK						
421			_VAT#0.5*(VA&VB),VBT#SQRT.(VA*VB)						
422			_VA#VAT,VB#VBT,NO#NO&1						
430			_ABS.((QROD-ROD1)/QROD)-DELTA		*		*		
431			_ABS.((SUN-SUM)/SUN)-DELTA		*		*		
440			_KNPH#QROD*PI*0.5 ,FK#KNPH&0.5*PI						
441			_DKEF(1)#FK                      \$K(K)						
442			_DKEF(3)#SUN*FK                      \$((E(K)-K(K))/K/K)						
443			_DKEF(2)#FK&ARG*ARG*SUN*FK      \$E(K)						
444			_DKEF(4)#KNPH                      \$K(K)-PI/2						
445			_DKEF(5)#NO						



## WIZ SOURCE PROGRAM

SEQ	LABL	TY	STATEMENT	C	ZE	NZE	PL	MI	ANY
500	_NEFF	_	_BETA#NEFF(1) \$KAPA NONZERO						
505	_	_	_KAPA#NEFF(2),KAPAP#NEFF(3) \$NEFF(4,5) DUMMY						
510	_	_	_VA#1,VB#KAPAP,BN#BETA,IF#0,NO1#0,TEMP#1						
515	*	_	_KP#VB/VA,PROD#TEMP,IFT#IF						
520	_	_	_DC#COS.(2*BN),NS#SIN.(2*BN)						
525	_	_	_BP#2*BN-ATAN.((1-KP)*NS/((1-KP)*DC&1&KP))						_15
530	_14	_	_BP#2*BN&ATAN.((1-KP)*(-NS)/((1-KP)*DC&1&KP))						_15
535	_15	_	_TEMP#PROD/(1&KP)*(BP/BN)						
540	_	_	_IF#BETA*TEMP						
545	_	_	_VAT#0.5*(VA&VB),VBT#SQRT.(VA*VB),NO1#NO1&1						
550	_	_	_VA#VAT,VB#VBT,BN#BP						
555	_	_	_ABS.((IF-IFT)/IF)-DELTA		*		*		
556	_	_	\$ IF COMPUTED						
560	_	_	DELO#KAPAP*KAPAP/(1&KAPA)						
565	_	_	_KK#1,MM#1,ENKS#(1-KAPA)*IF,FKK#IF*KAPA*KAPA						
566	_	_	_NO2#0						
570	_	_	_KN#KAPA,DELN#DELO,SSQ#SIN.(BETA),SSN#SSQ						
575	**	_	_SSQ#SSN*SSN,SSP#KN*KN*SSQ,RTKN#SQRT.(KN)						
580	_	_	_RT1#SQRT.(1-SSQ),RT2#SQRT.(1-SSP),ENKT#ENKS						
581	_	_	_RT3#SQRT.(0.5*(1&RT1)),RT4#SQRT.(0.5*(1-RT1))						
582	_	_	_RT5#SQRT.(0.5*(1&RT2)),RT6#SQRT.(0.5*(1-RT2))						
583	_	_	_AC#.5*(1&KN)*DELN*SSQ/((RT3&RT5)*(RT1&RT2))						
584	_	_	_AS# -(1&KN)*DELN*SSQ/((RT4&RT6)*(RT1&RT2))/2						
585	_	_	_SSH#RT4,CSH#RT3						
586	_	_	_DSN#AC*SSH&AS*CSH&DELN*SSN/(1&RTKN)						

## WIZ SOURCE PROGRAM

SEQ	_LABL_	_TY_	STATEMENT	_C_	ZE	_NZE_	PL	MI	_ANY
587	_	_	_KK#KK*KN,MM#2*MM						
590	_	_	_SSN#SSN&AC*SSH&AS*CSH, NO2#NO2&1						\$\$
591	_	_	_DELN#DELN*DELN/((1&KN)*(1&KN&2*RTKN))						
592	_	_	_KN#2*RTKN/(1&KN)						
593	_	_	_VN#-MM*DELN/(KK*KN)						
594	_	_	_UN#-MM*DSN/SQRT.(KK)						
595	_	_	_ENKS#ENKT&FKK*VN-KAPA*UN						\$ E-K*SIN.(BETA) _
596	_	_	_ABS.((ENKS-ENKT)/ENKS)-DELTA						_** _** _
597	_	_	_NEFF(1)#IF						\$ F(BETA,K#KAPA) _
598	_	_	_NEFF(2)#ENKS&KAPA*SSO						\$ E(BETA,K)
599	_	_	_NEFF(3)#ENKS,NEFF(4)#NO1,NEFF(5)#NO2						_ _ _ _ _
	_	_	\$ENKS IS E(BETA,K)-K*SIN(BETA),K#KAPA						
999	10	_	_END_						

APPENDIX IV. DERIVATION OF  $\Pi_N$ 

For  $\Pi\left(\frac{-2}{-1-\lambda_2}, g_2\right)$  the formulas 410.01 and 411.01 of Reference 12 are applicable, in which  $\alpha^2 = \frac{-2}{-1-\lambda_2} < 0$ ,  $k = g_2$ , i. e.,

$$\Pi_2 = \Pi(\alpha^2, k) = \frac{k^2 K(k)}{k^2 - \alpha^2} - \frac{\frac{\pi}{2} \alpha^2 \Lambda_0(\psi, k)}{\sqrt{\alpha^2(1-\alpha^2)(\alpha^2 - k^2)}} \quad [\text{IV-1}]$$

where

$$\psi = \sin^{-1} \sqrt{\frac{\alpha^2}{\alpha^2 - k^2}} \quad [\text{IV-1a}]$$

and

$$\Pi_2 = \frac{K(k)}{1-\alpha^2} + \frac{\frac{\pi}{2} \alpha^2 [\Lambda_0(\beta, k) - 1]}{\sqrt{\alpha^2(1-\alpha^2)(\alpha^2 - k^2)}} \quad [\text{IV-2}]$$

where

$$\beta = \sin^{-1} \frac{1}{\sqrt{1-\alpha^2}} \quad [\text{IV-2a}]$$

To exploit the possibility of gain in significant figures,  $\Pi_2 - \frac{K(k)}{1-\frac{\alpha^2}{2}}$  will not be computed by simple subtraction of  $\frac{K(k)}{1-\frac{\alpha^2}{2}}$  from either of the above equations. This difference is defined as  $\Pi_N$ .

From simple algebraic manipulation of Equations (IV-1) and (IV-2), one finds

$$\begin{aligned} \Pi_2 = \frac{K(k)}{1-\frac{1}{2}\alpha^2} + \frac{\frac{\pi}{2} \alpha^2}{\sqrt{\alpha^2(1-\alpha^2)(\alpha^2 - k^2)}} \left\{ (1-\alpha^2) [\Lambda_0(\beta, k) - 1] \left(1-\frac{1}{2}k^2\right) + \right. \\ \left. \frac{1}{2}(k^2 - \alpha^2) \Lambda_0(\psi, k) \right\} \frac{1}{(1-k^2)(1-\frac{1}{2}\alpha^2)} \end{aligned} \quad [\text{IV-3}]$$

For  $\rho_i, \rho_j$  small,  $\beta$  is near  $\frac{\pi}{2}$ ,  $k$  is near zero, thus  $\Lambda_o(\beta, k)$  is near but less than unity. Also for  $\rho_i, \rho_j$  near unity,  $k$  is near unity. Equation (IV-3) may still lose too many significant figures through  $\Lambda_o - 1$  or  $1 - k^2$ . One can further apply the addition formula (#153.01, Ref. 12) restricted to the condition that  $k \tan \beta \cdot \tan \psi = 1$ , i.e.,

$$\Lambda_o(\psi, k) + \Lambda_o(\beta, k) = 1 + \frac{2k'^2 \sin \beta \cos \beta K(k)}{\pi \sqrt{\cos^2 \beta + k'^2 \sin^2 \beta}} \quad [\text{IV-4}]$$

where  $k'^2 = 1 - k^2$

Eliminating  $\Lambda_o(\beta, k) - 1$  from (IV-4), (IV-1a), (IV-2a), one finds

$$\pi_2 = \frac{K(k)}{1 - \frac{\alpha^2}{2}} + \frac{1}{1 - \frac{\alpha^2}{2}} \left\{ \frac{\frac{\pi}{2} \alpha^2 (-1) (1 - \frac{1}{2} \alpha^2)}{\sqrt{\alpha^2 (1 - \alpha^2) (\alpha^2 - k^2)}} \Lambda_o(\psi, k) + \frac{\alpha^2 K(k) (1 - \frac{1}{2} k^2)}{(-\alpha^2 + k^2)} \right\} \quad [\text{IV-5}]$$

Therefore

$$\pi_N = \left( \pi_2 - \frac{K(k)}{1 - \frac{\alpha^2}{2}} \right) = \frac{-\alpha^2}{\sqrt{\alpha^2 (1 - \alpha^2) (\alpha^2 - k^2)}} \left\{ \frac{\pi}{2} (1 - \frac{1}{2} \alpha^2) \Lambda_o(\psi, k) - \frac{\sqrt{-\alpha^2} \sqrt{1 - \alpha^2}}{\sqrt{-\alpha^2 + k^2}} (1 - \frac{1}{2} k^2) K(k) \right\} \quad [\text{IV-6}]$$

Applying the addition formula (IV-4) again, one finds

$$\pi_N = -\alpha^2 \left\{ \frac{\pi}{2} \frac{(1 - \frac{\alpha^2}{2}) \Lambda_{id}}{\sqrt{-\alpha^2 (1 - \alpha^2) (k^2 - \alpha^2)}} - \frac{\frac{1}{2} K(k)}{(1 - \alpha^2)} \right\} \quad [\text{IV-7}]$$

$$L_{id} = (1 - \Lambda_0) = (1 - \sin \beta - (\Lambda_0 - \sin \beta))$$

$$= \frac{-\alpha^2}{\sqrt{1-\alpha^2} (1 + \sqrt{1-\alpha^2})} - L_{od} \quad \text{[IV-8]}$$

$$L_{od} = (\Lambda_0 - \sin \beta)$$

$$= \frac{2}{\pi} \left\{ [E(k) - F(k)] F(\beta, k') + K(k) E(\beta, k') - \frac{\pi}{2} \sin \beta \right\}$$

$$= \frac{2}{\pi} \left\{ k^2 D(k) F(\beta, k') + P(k) E(\beta, k') + \frac{\pi}{2} D(\beta, k') - \frac{\pi}{2} \frac{k^2}{(1+k')} \frac{1}{\sqrt{1-\alpha^2}} \right\} \quad \text{[IV-9]}$$

where

$$\alpha^2 = -2/(-1 - \lambda_2) \quad , \quad k = 2_2 \quad \text{[IV-9a]}$$

$$D(k) = DQ2 = (E(k) - F(k))/k^2 = DKEF(3) \quad \text{[IV-9b]}$$

$$P(k) = PQ2 = (K(k) - \frac{\pi}{2}) = DKEF(4) \quad \text{[IV-9c]}$$

$$D(\beta, k') = ID2 = (E(\beta, k') - k' \sin \beta) = NEFF(3) \quad \text{[IV-9d]}$$

There is apparently a gain of significant figures of  $\Pi_N$  when  $\rho_i - \rho_j$  are small ( $\alpha, k$  small) if equation (IV-7) is used, provided that the first term in the bracket can be evaluated as accurately as the second term. This is achieved by employing the subroutines DKEF and NEFF for equation (IV-9).

# APPENDIX V. X-FORCE ACTING ON THE TANK BY INTEGRATION OF PRESSURE

Assume a velocity potential

$$\phi_v = \dot{U} x + \sum_{n=1}^{\infty} a_n \phi_n(r, \psi, \theta) \quad [V-1]$$

where the first term is a particular solution satisfying the normal derivative condition on the sphere.  $\phi_n$  are the eigen functions which have no contribution to the normal velocity on the sphere. In order to satisfy the free surface condition for sinusoidal oscillations

$$\frac{\partial \phi_v}{\partial z} = \frac{\omega^2}{g} \phi_v \quad \text{or} \quad \frac{\omega^2}{g} \dot{U} x + \frac{\omega^2}{g} \sum_{n=1}^{\infty} \dot{a}_n \phi_n = \sum_{n=1}^{\infty} \dot{a}_n \frac{\omega_n^2}{g} \phi_n \quad \text{on } F$$

one has

$$\dot{a}_n = \frac{\frac{\omega_n^2}{\omega_n^2} \beta_n \dot{U}}{\left(\frac{\omega_n^2}{g} - \frac{\omega^2}{g}\right) a_n} = \frac{\frac{g}{\omega_n^2} \beta_n \dot{U}}{\left(\frac{\omega_n^2}{\omega^2} - 1\right) a_n} \quad [V-2]$$

since  $\phi_n$ 's are orthogonal on F,  $a_n = \int_F \phi_n^2 dS$  and  $\beta_n = \frac{\omega_n^2}{g} \int_F x \phi_n dS$

The pressure

$$p \cong -\rho_L \frac{\partial \phi_v}{\partial t} \cong -\rho_L \sum_{n=1}^{\infty} \ddot{a}_n \phi_n(r, \psi, \theta) - \rho_L \ddot{U} x \quad [V-3]$$

The x-force can be obtained by direct integration of pressure

$$\begin{aligned} F_x &= \int_R p \hat{i} \cdot \hat{n} dS = \rho_L \left\{ \int_V \text{div} \left[ -\hat{i} \ddot{U} x - \sum_{n=1}^{\infty} \ddot{a}_n \hat{i} \phi_n(r, \psi, \theta) \right] dV \right\} - \int_F p \hat{i} \cdot \hat{n} dF \\ &= -\rho_L V_L \ddot{U} - \rho_L \sum_{n=1}^{\infty} \ddot{a}_n \int_V \frac{\partial \phi_n}{\partial x} dV \\ &= -\rho_L V_L \ddot{U} - \rho_L \sum_{n=1}^{\infty} \ddot{a}_n \beta_n \end{aligned} \quad [V-4]$$

since

$$\begin{aligned}
 \beta_n &= \int_{V_L} \frac{\partial \phi_n}{\partial x} dV_L = \int_{V_L} [ \operatorname{div}(\chi \nabla \phi_n) - \cancel{\chi \nabla^2 \phi_n}^0 ] dV_L \\
 &= \int_F \chi \frac{\partial \phi_n}{\partial n_x} dS_x + \int_R \chi \frac{\partial \phi_n}{\partial n_x} dS_x \\
 &= \frac{\omega_n^2}{g} \int_F \chi \phi_n dS
 \end{aligned}$$

Equations [V-2, -4] are the same results as that obtained in

Reference 1 through Lagrangians' equations.



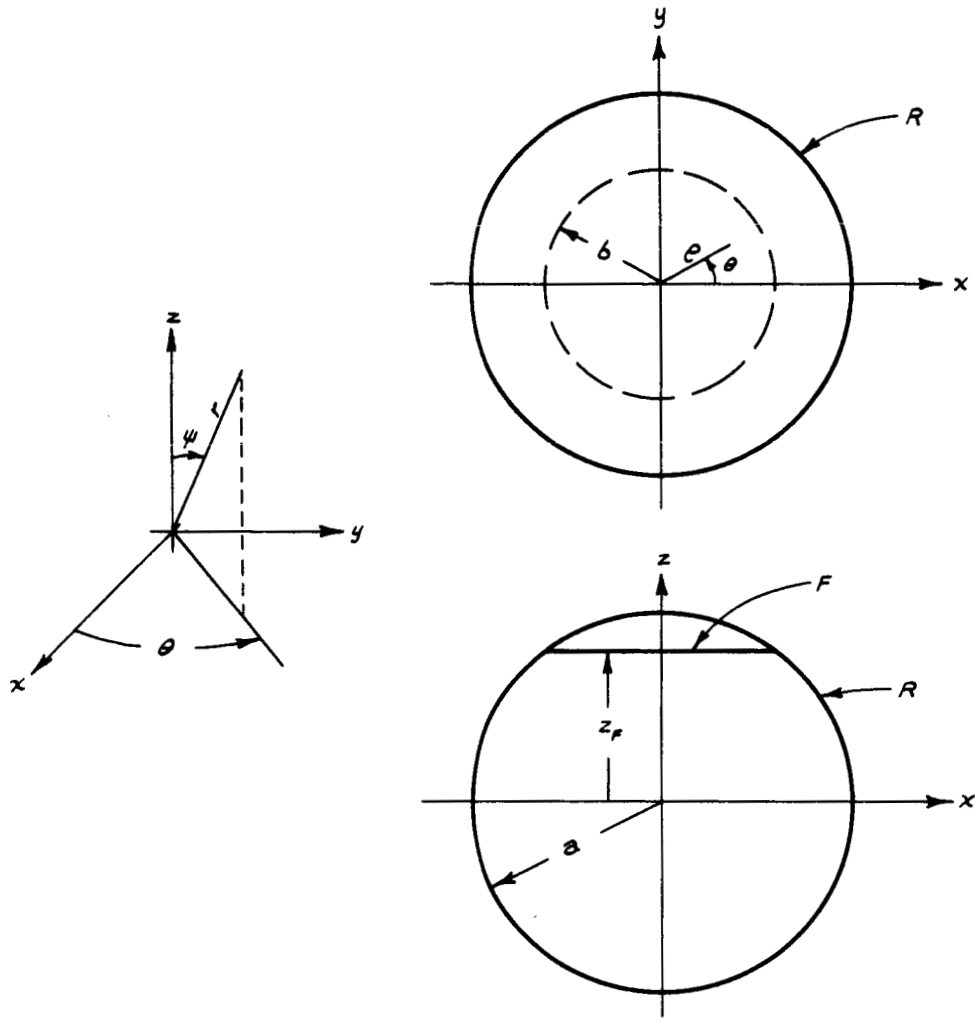


FIGURE 1. Graphical Illustration of Some Nomenclatures

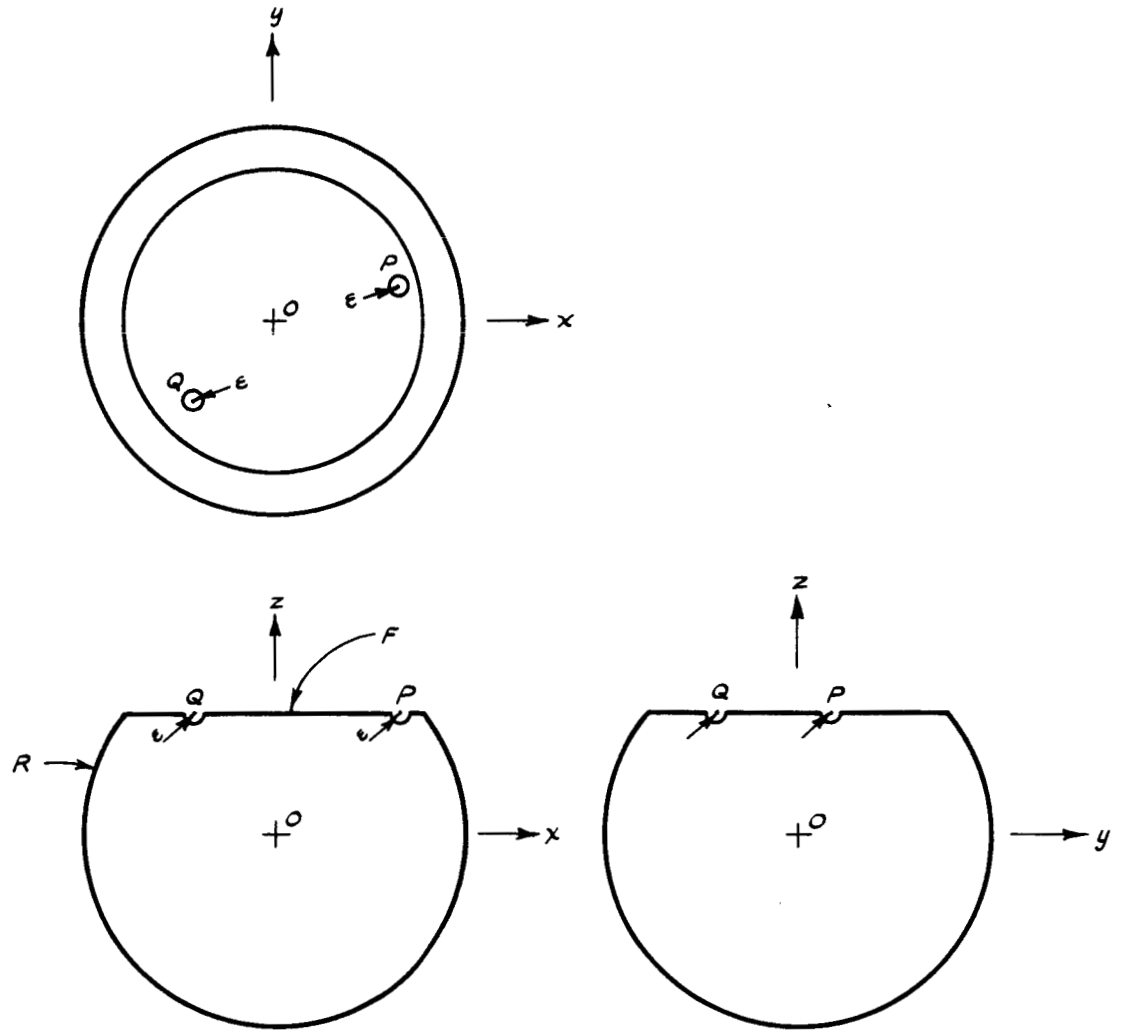


FIGURE 2. Surface of Integration

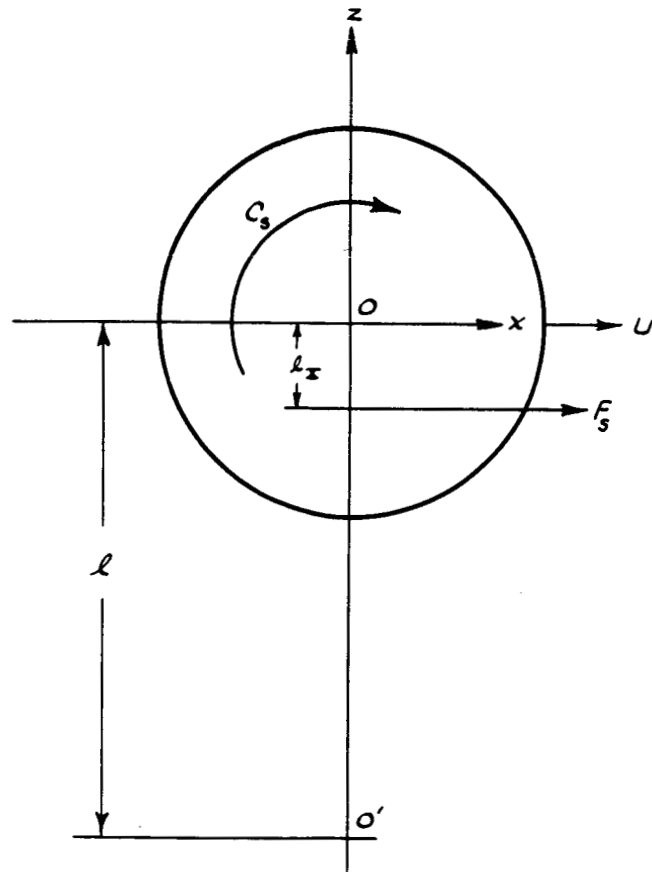


FIGURE 3a. Moment About Axis of Rotation

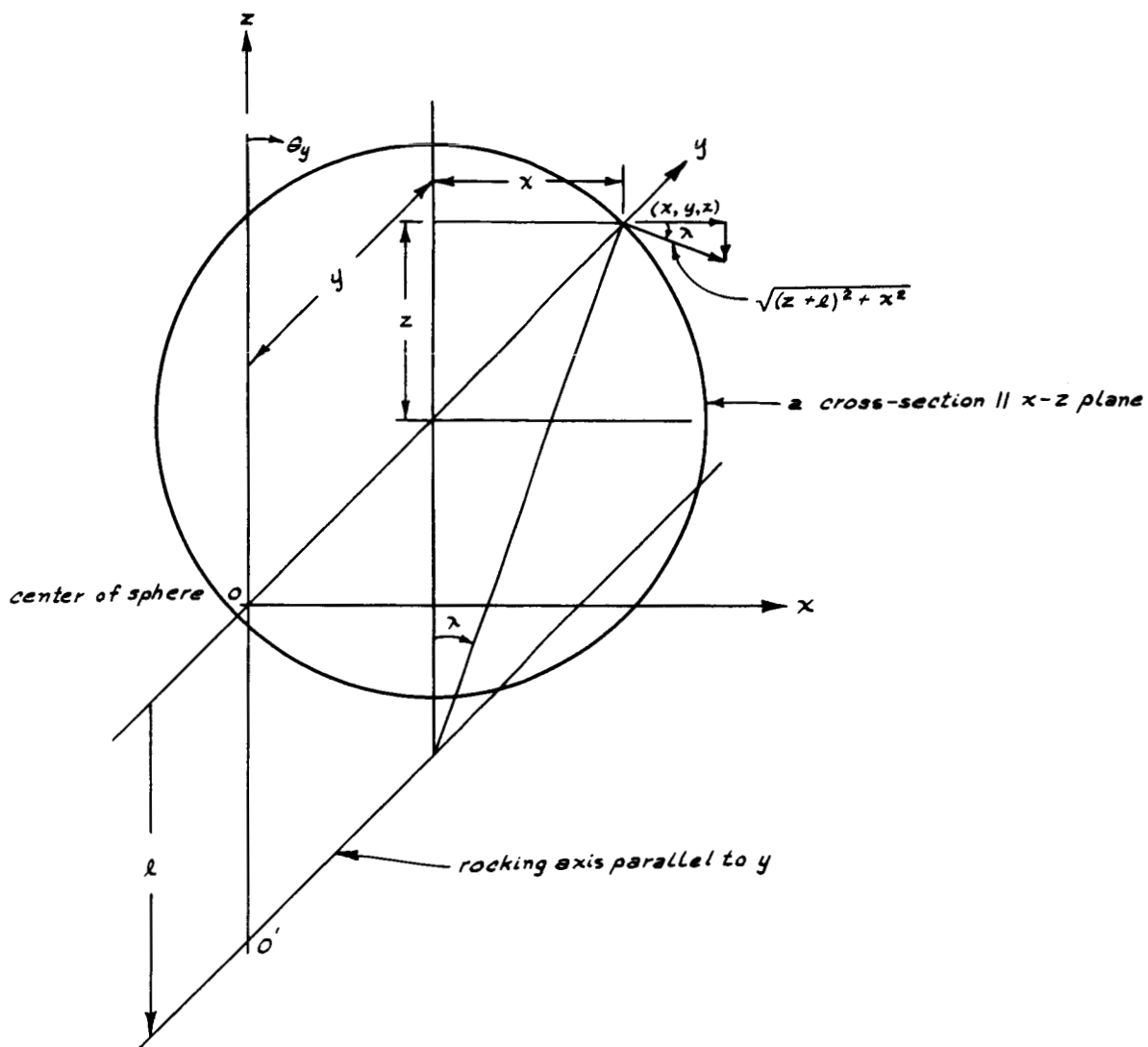


FIGURE 3b. Moment of Tank in Pitching (Rocking) Oscillation

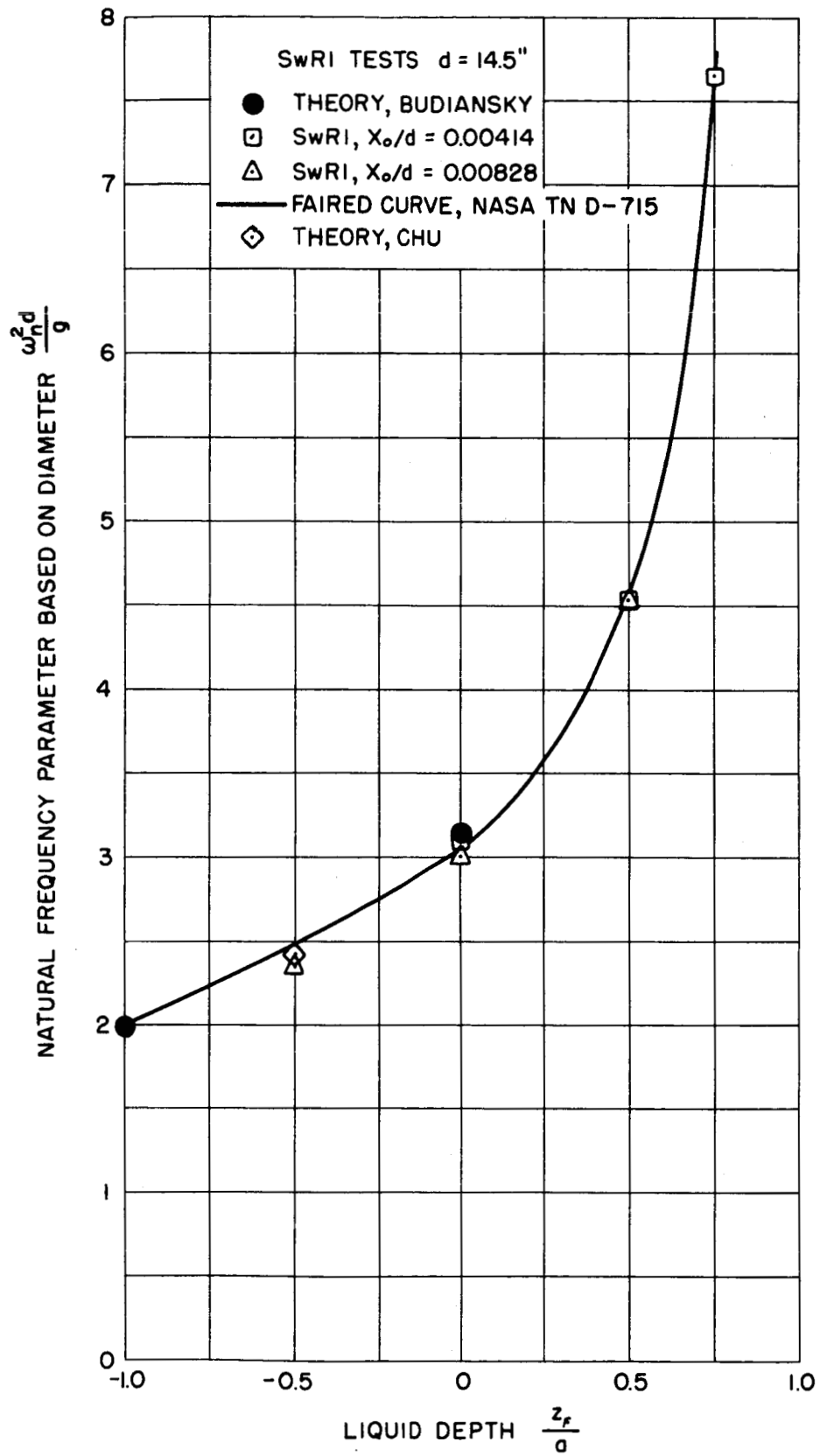


FIGURE 4a. Comparison of First Natural Frequency with Data by Abramson, et al., (Ref. 15)

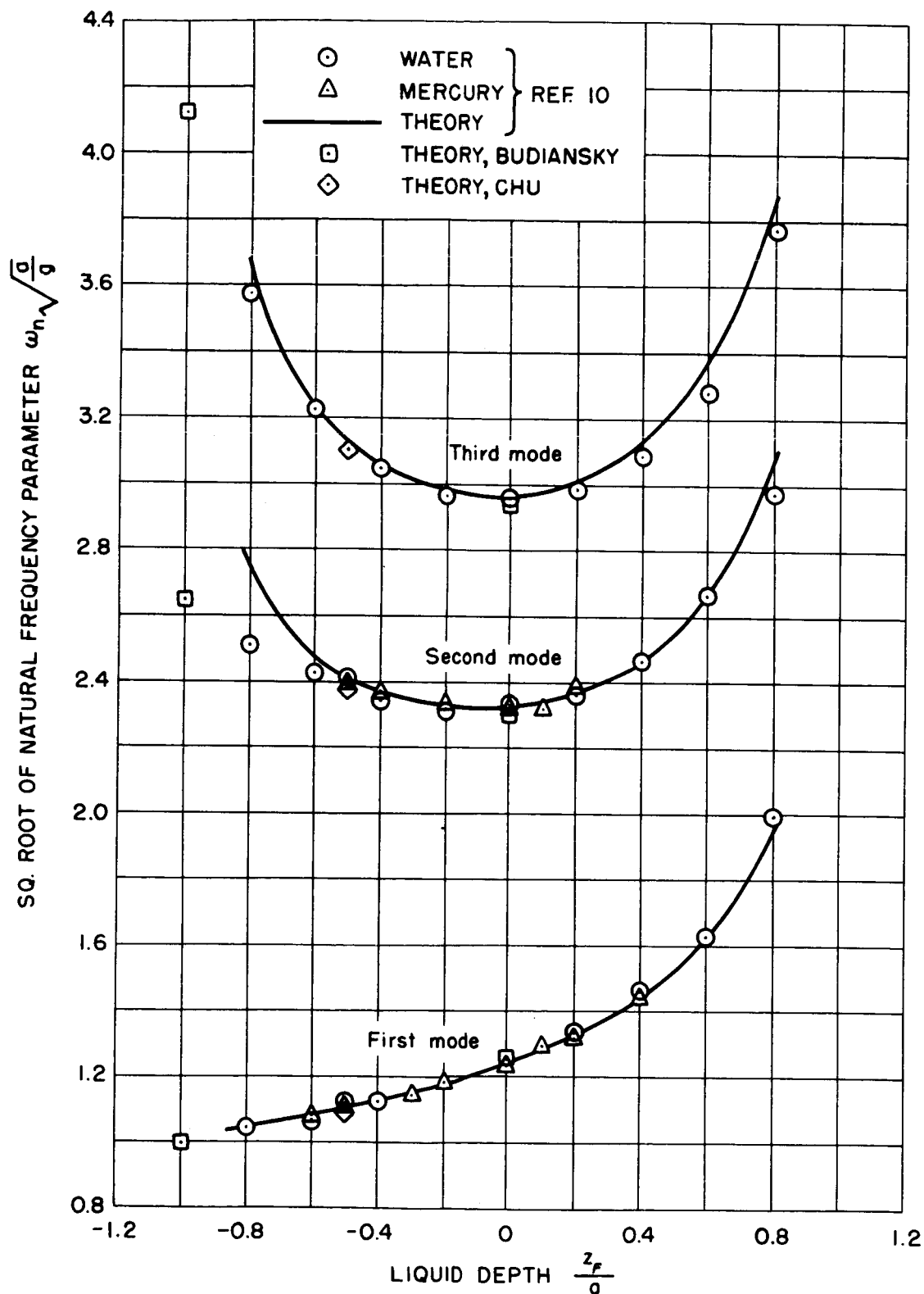


FIGURE 4b. Comparison of the First Three Natural Frequencies with Experiments of Stofan-Armstead (Ref. 10)

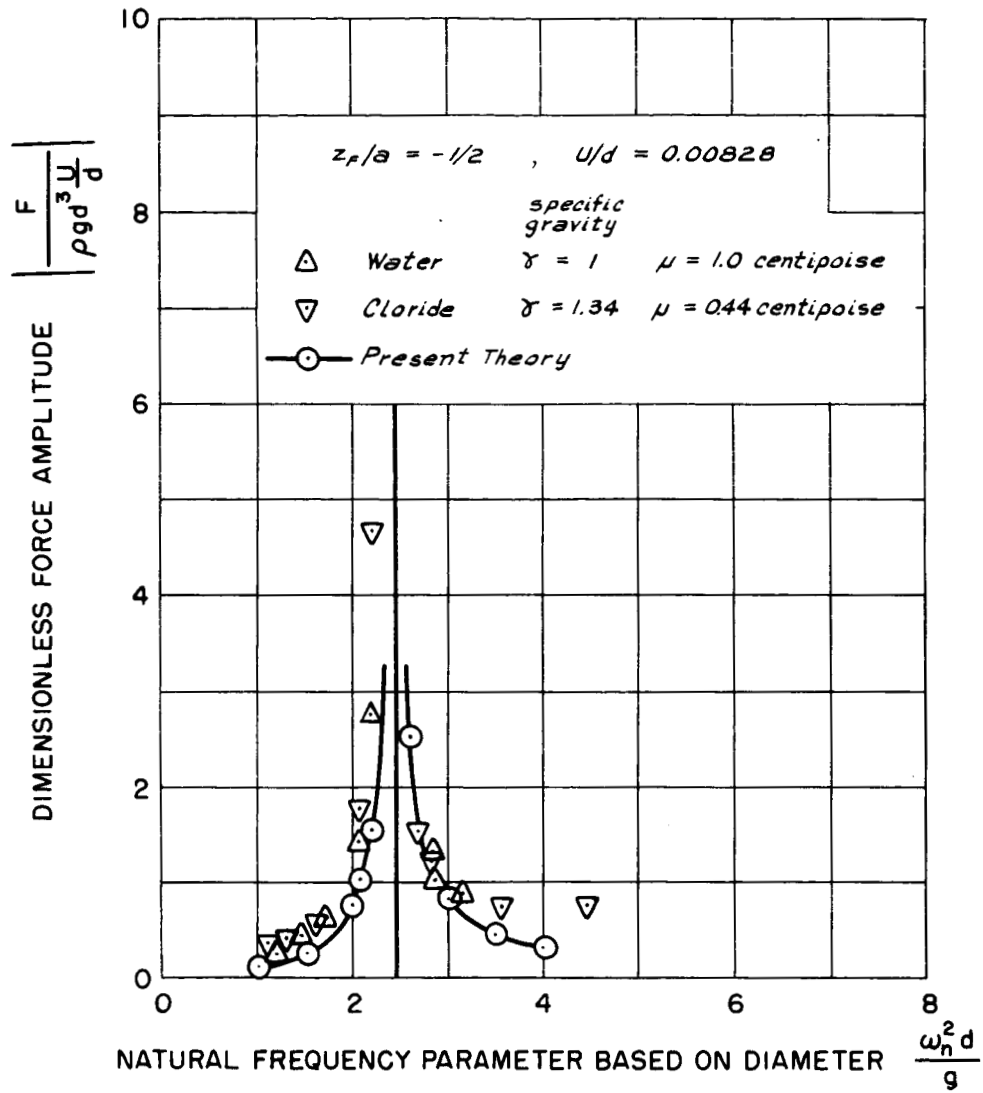


FIGURE 5. Comparison of Force Response for Quarter-Full Tank with Experiments by Abramson, et al., (Ref. 15)

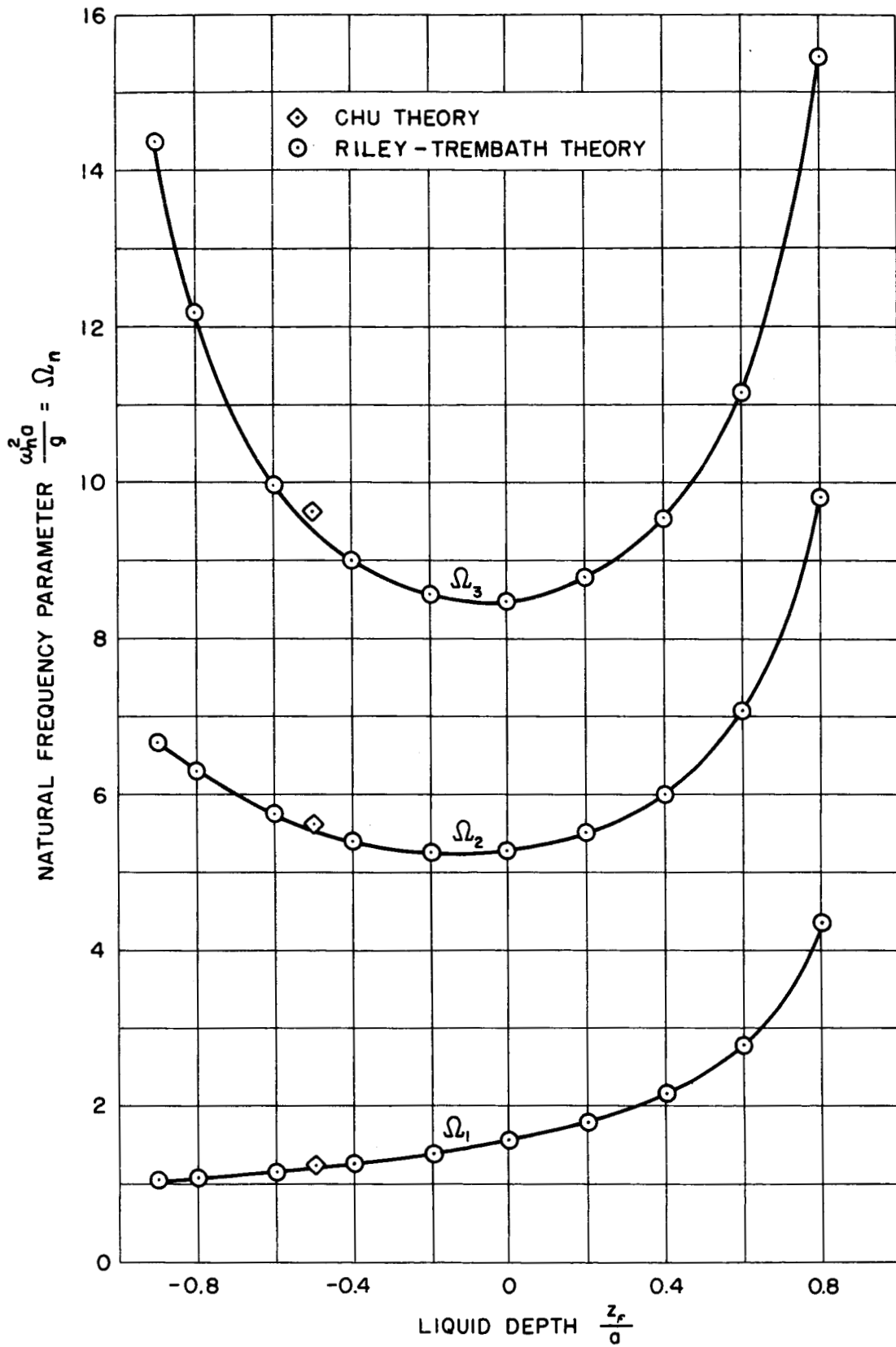


FIGURE 6. Natural Frequencies Given by Riley-Trembath (Ref. 17)